BLACK HOLES IN FIVE DIMENSIONS WITH $\mathbb{R} \times U(1)^2$ ISOMETRY

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This thesis is dedicated to the memory of

my brother, Chen Hui,

who left us in the winter of 2005, one month before his 22nd birthday, for all the love and care he had devoted, and the joy and fun he had brought to the family. Love, joy and peace in all of us.

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Summary

Black holes in higher space-time dimensions have been the subject of intensive study in the last decade, ever since the discovery of the black ring solution of Emparan and Reall. It is by now clear that higher-dimensional black holes have much richer structures than their four-dimensional counterparts. In this thesis we systematically study the simplest possible class of higher-dimensional black holes, i.e., vacuum black holes in five dimensions with $\mathbb{R} \times U(1)^2$ isometry, with a focus on the problem of classification and construction of these solutions.

For such a class of solutions, we first develop a stronger version of the rod-structure formalism than what has been previously used in the literature to analyze and classify them. In the asymptotically flat case, we then construct a new type of black holes—black lenses—with the last possible new horizon topology in five dimensions. The next step we put forward is to classify the spatial backgrounds of the class of black holes in five dimensions with $\mathbb{R} \times U(1)^2$ isometry, and we find that they are actually gravitational instantons, which were intensively studied in the literature 30 years ago, with $U(1) \times U(1)$ isometry. We then classify and construct black holes on such gravitational instantons, i.e., five-dimensional black holes whose spatial backgrounds are these gravitational instantons. At last we show that black holes

on the Taub-NUT instanton are equivalent to Kaluza–Klein black holes, if the latter are appropriately lifted to five dimensions.

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List of Symbols

Symbol	Definition
$\mathbb{Z}$	Integer numbers
$\mathbb{Z}_n$	Cyclic group of order n
$\mathbb{R}$	Real line; the field of real numbers
$\mathbb{C}$	Complex plane; the field of complex numbers
$\mathbb{R}^D$	$\{(x_1, x_2, ..., x_D) \vert x_i \in \mathbb{R}\}$
$\mathbb{R}P^n$	n-dimensional real projective space
$\mathbb{C}P^n$	n-dimensional complex projective space
$M^{1,D}$	$D+1$-dimensional Minkowski space-time
E^D	D-dimensional Euclidean space (with flat metric)
S^D	D-dimensional sphere
$U(1)$	$\{e^{i\theta} \vert \theta \in \mathbb{R}\}$
T^n	n-dimensional torus
$GL(2,\mathbb{Z})$	2 by 2 matrix group with integer entries and determinant ± 1
$SL(3,\mathbb{R})$	3 by 3 matrix group with unit determinant
$\lfloor x \rfloor$	The greatest integer no more than x

Symbol	Definition
D	Space-time dimension
G	Newton's constant
$g_{\mu\nu}$	Metric tensor
$R_{\mu\nu}$	Ricci tensor
$T_{\mu\nu}$	Energy-momentum tensor
R	Ricci scalar
J	Angular momentum
M	Mass
P	Magnetic charge
Q	Electric charge
S	Bekenstein–Hawking Entropy
T	Temperature
κ	Surface gravity
Ω	Angular velocity
∇	Covariant derivative operator
$\Box$	D'Alembert operator

Chapter 1

Introduction

1.1 Motivations to study black holes in higher dimensions

In the past decade, black holes in space-time dimensions $D \geq 5$ have been the subject of intensive study. There are a number of reasons to be interested in such a subject.

First of all, the idea that our space-time has extra dimensions is an indispensable ingredient in modern unifying theories, such as string/M theory, as well as some older contexts, such as Kaluza–Klein theory. In fact, in string/M theory, which is widely considered as the most promising "theory of everything" and, in particular, will describe quantum gravity, it is required that space-time has up to ten/eleven dimensions. One recent major achievement of string/M theory is that it explains the statistical origin of the Bekenstein–Hawking entropy of a five-dimensional black hole [1]. Higher-dimensional gravity and supergravity arise naturally as the low-energy effective theories in string/M theory. Understanding the former theories will help to gain insights to the full theory of the latter.

Secondly, the AdS/CFT correspondence [2, 3], or more generally the gauge/gravity correspondence, conjectured the equivalence between gravity in a bulk in certain dimensions and a quantum field theory defined on the boundary of the bulk, whose dimension is lower by one or more. Hence by this correspondence, higher-dimensional gravity can be mapped to describe certain lower-dimensional quantum field theories.

Thirdly, in braneworld scenarios [4] or TeV gravity [5–7], it has been predicted that microscopic higher-dimensional black holes might be produced and detected at the LHC. In these scenarios, to resolve the hierarchy problem, it is assumed that there exist large extra dimensions. This allows for the experimental determination of a number of theoretical assumptions or predictions, such as the fundamental scale of gravity, the number of extra dimensions, etc.

And last but not least, black holes in higher dimensions deserve study in their own right. Even if our space-time eventually turns out to have only four dimensions, we might be asked the more fundamental question, "Why four?". The answer cannot be found unless we know what really happens and what goes wrong in higher dimensions. By taking the space-time dimension, in the theory of gravity, as a tunable parameter, we will be able see what are peculiar to four dimensions, and what are universal for all dimensions. Black holes are among the most interesting objects in general relativity, and, of course, deserve study.

The above are just a few among many of the motivations to study higher-dimensional black holes. Personally, I am more motivated to study them from a mathematical perspective: I got excited when I calculated the Ricci tensors and found that they are zero for the vacuum black holes/gravitational instantons that will be studied in this thesis! In what follows we will give a brief review of the current status of vacuum black holes in higher dimensions.

1.2 Richer structures of black holes in higher dimensions

In four space-time dimensions, it is well known that stationary, asymptotically flat vacuum black holes are uniquely determined by the asymptotic charges, i.e., the mass and angular momentum, and their only allowed horizon topology is S^2. In fact, they must coincide with the unique solution found by Kerr [8]. This is widely known as the uniqueness theorem of black holes in four dimensions [9–14]. This result excludes the possibility of a four-dimensional black hole with other horizon topologies, such as $S^1 \times S^1$. It also excludes the possibility of a multi-black hole configuration in equilibrium in four dimensions. These states, if they exist, cannot be stable, and they must evolve to a stationary final state described by the Kerr solution. For more aspects of black hole solutions in four dimensions, see, e.g., [15–17] for reviews.

1.2.1 Black holes in higher dimensions $D \geq 5$

In higher dimensions, it has been recently found that, in contrast to four dimensions, black holes exhibit much richer and more complicated phase structures. In particular, non-spherical horizon topologies are possible, and the uniqueness theorem is violated. This can best be seen in five asymptotically flat space-time dimensions, as demonstrated by the Myers–Perry black hole [18] and the recently discovered Emparan–Reall black ring [19]. These two types of black holes, with rather different horizon topologies S^3 and $S^1 \times S^2$ respectively, can in certain cases carry the same mass and angular momentum. The reader is referred to [20–22] and references therein for more detailed reviews on the rich phase structures of black holes in higher dimensions.

Some obvious reasons are responsible for the complicated structures of black holes in higher dimensions. Firstly, as the number of dimensions D grows, the number of independent axes, along which the black holes can rotate, grows. This means that the black holes can carry more independent rotational parameters, so there are now more degrees of freedom for their dynamics. Secondly, in higher dimensions, there exist various extended black objects such as black strings/rings/branes. The restrictions of the topologies of black objects in higher dimensions are, generally speaking, rather loose. Thirdly, higher space-time dimensions allow for various possible compact directions, e.g., there may exist bubbles or NUT charges. These space-times, though completely regular, are not asymptotically flat. Black holes in these space-times have even more complicated phase structures [21].

In asymptotically flat space-times, the black hole of Myers and Perry is the natural generalization of the Kerr black hole to arbitrary dimensions $D \geq 5$. It has a spherical horizon topology S^{D-2}, and is rotating with $\lfloor \frac{D-1}{2} \rfloor$ independent angular momenta along all possible asymptotic axes. Up to date, in $D > 5$, the Myers–Perry black hole is still the only explicitly known analytic asymptotically flat vacuum solution. Black rings with horizon topology $S^1 \times S^{D-3}$, or more general types of back objects known as blackfolds, have been constructed in any dimensions $D \geq 5$ [23–25], but all in perturbation theories. Major breakthroughs on exact black hole solutions in dimensions $D \geq 6$ can be foreseen in the future.

We also review here some other relevant aspects of black holes in any asymptotically flat space-time dimensions $D \geq 5$. First of all, the possible black hole horizon topologies have been classified [26–28] and are shown to be of positive Yamabe type [28], i.e., admit metrics of positive scalar curvature. Secondly, as the static limit of the Myers–Perry black hole, the higher-dimensional Schwarzschild black hole [29] is proved to be the unique solution in static space-times [30, 31]. Hence, in the static regime, the structures of higher-dimensional asymptotically flat black holes are still rather simple. Thirdly, for stationary black holes, the rigidity theorem has been established [32–34], which guarantees the existence of a $U(1)$ isometry subgroup for such black holes.

There is a particular class of black holes in any space-time dimensions $D \geq 4$ that is more tractable to mathematical analysis and has been studied extensively, namely stationary vacuum black holes with non-degenerate horizons, admitting an additional $D-3$ mutually commuting space-like Killing vector fields (with closed orbits) [35–40]. For a given solution in such a class, the rod structure has been defined, which turns out to be a very useful tool to analyze and characterize the solution. These studies are higher-dimensional generalizations of the four-dimensional case previously studied by Weyl [41] and Papapetrou [42, 43]. Powerful solution-generating techniques, such as the inverse scattering method [44–47], have also been developed to construct new types of solutions within this class. We note that solutions within this class can be asymptotically flat only in the case when $D = 4, 5$. This is because the isometry group of asymptotically flat space-times in D dimensions allows for at most a Cartan subgroup $U(1)^{\lfloor \frac{D-1}{2} \rfloor}$. We thus require that $U(1)^{D-3}$ is a subgroup of $U(1)^{\lfloor \frac{D-1}{2} \rfloor}$, which eventually leads to $D = 4, 5$.

1.2.2 Black holes in five dimensions

For black holes in dimension $D = 5$, more concrete and complete results have been obtained in the past decade.

For asymptotically flat stationary black holes, the rigidity theorems [32–34] guarantee that the full isometry group of these black holes is at least $\mathbb{R} \times U(1)$. The black holes with exactly the isometry group $\mathbb{R} \times U(1)$ were first conjectured in [48], but up to date, none of them are explicitly known. We note, however, Emparan et al. claimed they have constructed this type of black holes (helical black rings) using approximation methods [24].

If we assume an additional $U(1)$ isometry, such that the isometry group of the black holes is now $\mathbb{R} \times U(1)^2$, many results have been obtained so far. Firstly, all possible black hole horizon topologies have been classified by Hollands and

Yazadjiev [38] using the rod structure formalism.[1] These black holes, if realized, are also proved to be unique and are specified by their angular momenta and rod structure. Secondly, using the inverse scattering method, many exact black hole solutions have been constructed, which have been found to exhibit a very rich phase structure. Among these black hole solutions are the Emparan–Reall black ring with single angular momentum [19] and Pomeransky–Sen'kov black ring with two independent angular momenta [49], the black saturn [50], the black di-ring [51, 52], and the black bi-ring [53, 54]. The phase structure of these black holes in five dimensions have been studied very thoroughly, see [20] for a review.

Black holes in space-times that asymptote to a direct product $M^{1,3} \times S^1$ have also been studied. If we assume again the isometry group $\mathbb{R} \times U(1)^2$, the black hole horizon topologies have been classified and uniqueness theorems have been proved [39]. Many exact solutions have also been constructed [35, 55–59]. Furthermore, various solutions in perturbation theories have also been constructed, see the review [21].

Another class of solutions, known as squashed Kaluza–Klein black holes [60, 61], have attracted considerable attention recently. Their asymptotic geometry is a non-trivial S^1 fiber bundle over $M^{1,3}$, and they also possess an isometry group $\mathbb{R} \times U(1)^2$. Black rings within this class have also been constructed [62, 63].

[1] Hollands and Yazadjiev in [38, 39] used the terminology "interval structure" instead. The relations between the interval structure and the rod structure of a solution will be discussed in detail in chapter 3.

1.3 Scope and organization

In this thesis, we will systematically study stationary vacuum black hole solutions in five dimensions with $\mathbb{R} \times U(1)^2$ isometry, with a focus on the problem of classification and construction of these solutions. We emphasize that we do not presume the type of their asymptotic geometries. These black hole space-times are solutions to the vacuum Einstein field equations with zero cosmological constant, i.e., $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$, which in fact reduce to the Ricci-flat equations, i.e., $R_{\mu\nu} = 0$.

We will not discuss the stability properties of these black holes, though it is generally true that black holes in higher dimensions with a string/brane-like horizon shape which has some much more extended directions than others and may be caused by fast rotations, will suffer from the Gregory–Laflamme instability [64, 65]. Black holes in higher dimensions in other contexts, e.g., with a non-vanishing cosmological constant [66, 67], in minimal supergravity [68–70], or with other matter sources such as dipole charges [71, 72], will not be considered. Approximation and numerical methods are also beyond the scope of this thesis.

This thesis is organized as follows. We will first show that, for black holes in five dimensions with $\mathbb{R} \times U(1)^2$ isometry, regardless of their asymptotic geometries, we can define the rod structure to analyze and characterize them. This is done in the first section of chapter 3, the material of which is based on part of our paper [73]. The rod structure formalism that will be developed can be regarded as an extension of the rod structure of [38, 39] to arbitrary asymptotic geometries, and can also be regarded as a stronger version of the rod structure of [36, 37] by taking into consideration the global properties of space-time structure. The most powerful solution generating technique that has been applied in the literature to generate five-dimensional black holes with $\mathbb{R} \times U(1)^2$ isometry, i.e., the inverse scattering method, will be reviewed in the second section of chapter 3.

In asymptotically flat space-times, the possible horizon topologies of black holes with $\mathbb{R} \times U(1)^2$ isometry are shown to be either S^3, $S^1 \times S^2$, or a lens space $L(p, q)$ for some coprime integers p and q [38]. We will consider the third possibility and construct the so-called black lens solutions with horizon topology $L(n, 1)$ in chapter 4, the material of which is based on our papers [74, 75].

In the effort to classify and construct black holes with $\mathbb{R} \times U(1)^2$ isometry, one may first try to classify the possible spatial backgrounds of these space-times. We find that gravitational instantons with $U(1) \times U(1)$ isometry can serve as these possible spatial backgrounds. The rod-structure formalism then naturally provides a scheme to classify these gravitational instantons. This is done in chapter 5, the material of which is based on our paper [73].

The gravitational instantons with $U(1) \times U(1)$ isometry have various asymptotic geometries other than just E^4 or $E^3 \times S^1$. We can add a flat time dimension to them to obtain five-dimensional space-times with $\mathbb{R} \times U(1)^2$ isometry, as solutions to vacuum Einstein equations. Moreover, we can add stationary black holes to such space-times while preserving the $U(1) \times U(1)$ isometry. The black hole space-times thus obtained have various asymptotic geometries other than just $M^{1,4}$ or $M^{1,3} \times S^1$. This is done in chapter 6, the material of which is based on our paper [76].

Black holes constructed on the self-dual Taub-NUT instanton have very interesting interpretations in Kaluza–Klein theory. In fact, it will be shown in chapter 7 that they are equivalent to the solutions by appropriately lifting the Kaluza–Klein black holes to five dimensions. The material of this chapter is based on our unpublished draft [77].

For completeness, we review in chapter 2 some well-known black hole solutions in four and five dimensions. This thesis ends with a brief discussion on our results and some open problems in chapter 8.

Chapter 2

Review of some known black holes

In this chapter, we review some well-known black hole solutions in four and five asymptotically flat space-time dimensions. These include the Kerr black hole, five-dimensional Myers–Perry black hole, and Emparan–Reall black ring. For the purpose of this thesis, we focus on the solutions themselves. All of these solutions have the prescribed isometries for us to define their rod structures, which will be discussed in detail in the next chapter. We also briefly discuss their physical quantities, for the reader to understand the phases of these black holes. These black holes are well-known and have been studied extensively in the literature; for more reviews on their various aspects, the reader is referred to [17, 18, 20, 78].

2.1 Kerr black hole

The metric of the Kerr black hole [8] in Boyer–Lindquist coordinates is described by

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{\Delta\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi\right)^2}{\Sigma} + \frac{\sin^2\theta\,\left[a\mathrm{d}t - (r^2+a^2)\,\mathrm{d}\phi\right]^2}{\Sigma} \\ &\quad +\Sigma\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right), \end{aligned} \tag{2.1}$$

where Σ and Δ are defined as

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2mr + a^2. \tag{2.2}$$

This metric describes a rotating black hole with a spherical event horizon S^2 located at $r = r_0 \equiv m + \sqrt{m^2 - a^2}$. The ADM mass M and angular momentum J of this black hole are $M = m$ and $J = ma$ respectively. When the angular momentum parameter $a = 0$, we recover its static limit, namely, the Schwarzschild black hole. For fixed mass M, when a and so the angular momentum increases, the area of horizon (and so the entropy) of this black hole decreases. The angular momentum parameter a is bounded from above by $|a| = m$, in which case the Kerr black hole becomes extremal with a non-singular horizon with a minimum but finite area.

In the solution (2.1), ϕ is an azimuthal coordinate, parameterizing the axial symmetry $U(1)$ of the space-time, and it has the normal period 2π. The isometry group of this solution is then $\mathbb{R} \times U(1)$, where $\mathbb{R}$ corresponds to the flow of time. It can be checked that, at infinity $r \to \infty$, the above solution approaches the Minkowski space-time $M^{1,3}$. The uniqueness theorem [9–14] asserts that the Kerr black hole (2.1) is the unique solution in four-dimensional asymptotically flat vacuum space-times. Hence the Kerr solution exhausts all the possible solutions in four flat dimensions.

2.2 Five-dimensional Myers–Perry black hole

The natural generalization of the Kerr black hole to arbitrary higher space-time dimensions was found by Myers and Perry in 1986 [18]. This black hole presents many new interesting features, as well as many similar to that of the Kerr black hole [20]. In the case of five dimensions, the black hole is described by the metric

$$\begin{aligned} \mathrm{d}s^2 &= \frac{2m}{\Sigma}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\psi - b\cos^2\theta\,\mathrm{d}\phi\right)^2 - \mathrm{d}t^2 + \Sigma\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right) \\ &\quad + \left(r^2 + a^2\right)\sin^2\theta\,\mathrm{d}\psi^2 + \left(r^2 + b^2\right)\cos^2\theta\,\mathrm{d}\phi^2, \end{aligned} \tag{2.3}$$

where Σ and Δ are defined as

$$\Sigma = r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \qquad \Delta = \frac{(r^2+a^2)(r^2+b^2)}{r^2} - 2m. \tag{2.4}$$

This metric describes a rotating black hole in five space-time dimensions with a spherical horizon topology S^3. The event horizon is located at the greatest root of Δ, i.e., $r = r_0 \equiv \frac{1}{2}\left[\sqrt{2m-(a-b)^2} + \sqrt{2m-(a+b)^2}\right]$. To ensure this root to be real, the parameters must satisfy the condition $|a| + |b| \leq \sqrt{2m}$. As is well known, in five asymptotically flat space-time dimensions, a black hole can rotate independently along two orthogonal rotational axes [18]. The five-dimensional Myers–Perry black hole has an ADM mass $M = \frac{3\pi}{4}m$, and two independent angular momenta $J_1 = \frac{\pi}{2}ma$ and $J_2 = \frac{\pi}{2}mb$, along the two rotational axes parameterized by the azimuthal coordinates ψ and ϕ respectively. For fixed mass M, it is obvious that the angular momenta are bounded by the relation $|J_1| + |J_2| \leq \sqrt{\frac{32M^3}{27\pi^2}}$. When this inequality is saturated, the black hole will have regular horizons if and only if both of the two angular momenta are non-vanishing. When both angular momenta are zero, we recover the five-dimensional Schwarzschild black hole [29].

The two asymptotic axes are parameterized by the two azimuthal coordinates ψ and ϕ. It will be shown later that, to ensure asymptotic flatness, these two coordinates must have period 2π independently. The isometry group of the five-dimensional Myers–Perry black hole is then $\mathbb{R}\times U(1)^2$, where $\mathbb{R}$ corresponds to the flow of time, and the two $U(1)$'s correspond to the two rotational axial symmetries. At infinity $r \to \infty$, it approaches the five-dimensional Minkowski space-time $M^{1,4}$. One might ponder whether the solution (2.3) is the unique solution in five asymptotically flat space-time dimensions, specified by the mass and angular momenta. The answer to this question is no, with a counterexample given by the black ring solution discovered by Emparan and Reall, which will be reviewed in the following section.

2.3 Emparan–Reall black ring

A novel black hole solution in five space-time dimensions was found by Emparan and Reall in 2001 [19]. It has a ring-shaped horizon topology $S^1\times S^2$, and, in certain cases, can carry the same mass and angular momentum as the five-dimensional Myers–Perry black hole. So the black hole uniqueness theorems in four dimensions do not simply generalize to higher dimensions. In fact, Emparan and Reall found a general black ring solution with rotation along the S^1 direction, that may not necessarily be balanced. This solution has a metric [19, 20, 78]

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{F(y)}{F(x)}(\mathrm{d}t+\Omega)^2 + \frac{2\varkappa^2 F(x)}{(x-y)^2} \\ &\quad \times\Big[-\frac{G(y)}{F(y)}\mathrm{d}\psi^2 + \frac{G(x)}{F(x)}\mathrm{d}\phi^2 + \frac{(1-c)^2}{1-b}\left(\frac{\mathrm{d}x^2}{G(x)} - \frac{\mathrm{d}y^2}{G(y)}\right)\Big], \end{aligned} \tag{2.5}$$

where the one-form Ω is defined as

$$\Omega = -\varkappa\sqrt{\frac{2b(b-c)(1+b)}{1-b}}\frac{(1+y)}{F(y)}\,\mathrm{d}\psi, \tag{2.6}$$

and the functions $F(\xi)$ and $G(\xi)$ are given by

$$F(\xi) = 1 + b\xi, \qquad G(\xi) = (1+c\xi)(1-\xi^2). \tag{2.7}$$

The parameters c, b, and the coordinates x, y take the ranges $0 \le c \le b < 1$, $-1 \le x \le 1$, and $-\frac{1}{c} \le y \le -1$.

In general, the solution (2.5) describes a black hole with horizon topology $S^1 \times S^2$ rotating along the S^1 direction in the presence of a conical singularity. In this configuration, the rotation provides a centrifugal force against the self-attraction of the black hole. The presence of a conical singularity (which may have a deficit or excess angle) is then balancing the whole system. In the special case when the conical singularity is absent, we get a regular black ring, with rotation alone balancing self-gravitation. This occurs when $b = \frac{2c}{1+c^2}$.

The coordinates ψ and ϕ parameterize the two asymptotic axes of the space-time. One of these axes, parameterized by ψ, is located at $x = -1$. It extends from infinity ($x \to -1, y \to -1$), and reaches the black ring horizon. Passing through the black ring, there is an inner axis $x = 1$ (parameterized by ψ), which meets at the center of the black ring's S^1 with another axis $y = -1$ parameterized by ϕ, which extends to infinity. The black ring is rotating along the ψ direction, and has an event horizon located at $y = -\frac{1}{c}$. It will be clear later that to ensure asymptotic flatness, the two coordinates ψ and ϕ should have period 2π independently. Together with the time coordinate t, they parameterize the isometry group $\mathbb{R} \times U(1)^2$ of the Emparan–Reall black ring, similar to that of the five-dimensional Myers–Perry black hole discussed in the previous section.

The ADM mass and angular momenta of this black ring solution are

$$M = \frac{3\pi\varkappa^2 b(1-c)}{2(1-b)}, \quad J_\psi = \frac{\pi\varkappa^3\sqrt{2b(1+b)(b-c)}(1-c)}{(1-b)^{3/2}}, \quad J_\phi = 0. \tag{2.8}$$

For fixed mass M, the angular momentum of a regular black ring is bounded from below by $|J| \geq \sqrt{\frac{M^3}{\pi}}$; in particular, the angular momentum can be arbitrarily large. For certain ranges of parameters, the Emparan–Reall black ring can carry the same mass and angular momentum as the five-dimensional Myers–Perry black hole [19]. This provides a counterexample of a possible generalization of the uniqueness theorems in four dimensions to higher dimensions. Moreover, we notice that, even when restricted to horizon topology of $S^1 \times S^2$, black holes in five dimensions cannot be uniquely determined by their asymptotic quantities, as in certain cases, two different branches of black rings can share the same mass and angular momentum!

The Emparan–Reall black ring carries a single angular momentum along its S^1 direction parameterized by ψ. Recall that a black hole in five dimensions can in general rotate along the two orthogonal asymptotic axes. The Emparan–Reall black ring was later generalized by Pomeransky and Sen'kov [49] to a double-rotating black ring with another angular momentum along the azimuthal angle in S^2 parameterized by ϕ.

Chapter 3

Analyzing methods and solution-generating techniques

Physicists often found exact solutions in the form of local metrics before any interpretations can be made. These metrics, of course, describe the local space-time geometries, related to the local matter distributions through Einstein field equations. However, these metrics may not be able to describe a well-behaved global space-time. This is because when one tries to define a global space-time by gluing together small pieces described by local metrics, some pathologies might appear. An example was given by Misner [79], who tried to interpret the Taub-NUT metric and found that it is necessary for the space-time to have closed time-like curves (CTCs). It is generally unknown whether a metric can be interpreted as a global space-time and what the necessary conditions are. However, for black hole solutions in five dimensions with $\mathbb{R} \times U(1)^2$ isometry, we will show in section 3.1 that the rod-structure formalism provides a suitable tool for the analysis.

An equally fundamental problem is how to find the local metrics which solve Einstein equations. Physicists have developed powerful solution-generating techniques to construct solutions by starting with some known ones. We can then avoid solving Einstein field equations directly by applying these solution-generating techniques. We will discuss in section 3.2 the inverse scattering method (ISM), which has been shown to be very powerful in generating five-dimensional solutions with $\mathbb{R} \times U(1)^2$ isometry.

3.1 Rod-structure analysis

We begin by considering the class of stationary vacuum black holes in D space-time dimensions ($D \geq 4$) with non-degenerate horizons, admitting an additional $D-3$ mutually commuting space-like Killing vector fields (with closed orbits). The symmetries corresponding to these Killing vector fields are referred to as "axial symmetries", even though in the general case for $D > 4$, their fixed-point sets are higher-dimensional surfaces rather than a real axis as in $D = 4$. For a static black hole space-time with an additional $D-3$ orthogonal space-like Killing vector fields, it turns out that the Einstein equations decouple into two sets. One of them resembles a three-dimensional flat-space Laplace equation, and the solutions correspond to rod-like sources along a line in the three-dimensional space [35]. This formalism was subsequently generalized to stationary black hole space-times by Harmark et al. [36, 37].

Thus each black hole solution in this class will have a certain so-called rod structure associated to it, with the rods themselves physically representing either the event horizon or the symmetry axes. Much information can be read off from a given rod structure, for example, the topology of the event horizon and certain asymptotic properties of the space-time. Recently, there have also been some attempts to use the rod-structure formalism to extend the four-dimensional black hole uniqueness theorems to higher dimensions. By defining a more mathematical version of the rod structure (known as the interval structure) that takes into account the global properties of the space-time, Hollands and Yazadjiev [38, 39] proved certain uniqueness theorems for stationary black holes which are either asymptotically $M^{1,D-1}$, or asymptotically $M^{1,s} \times T^{D-s-1}$ where $0 < s < D-1$ (see also [40] for more aspects of these space-times).

In this thesis, we are interested in this special class of solutions in five space-time dimensions, in particular, those whose two space-like Killing vector fields generate closed orbits. So the isometry group of these solutions is $\mathcal{G} = \mathbb{R} \times \mathcal{T}$, where $\mathbb{R}$ corresponds to the flow of time, and $\mathcal{T} = U(1) \times U(1)$ corresponds to the flows of the two space-like Killing vector fields.[1] Well-known solutions belonging to this class include the five-dimensional Myers–Perry black hole and the Emparan–Reall black ring.

The aim of this section is to describe a way in which the possible conical and

[1]It is a theorem that every compact, connected, two-dimensional Lie group is commutative, and therefore isomorphic to $U(1) \times U(1)$. This makes it clear why the two commuting space-like Killing vector fields with closed orbits generate an (effective) $U(1) \times U(1)$ isometry group action for the space(-times) considered in this thesis.

orbifold singularities of the black hole space-times can be readily read off from the rod structure. To do so, we first introduce in subsection 3.1.1 a stronger version of the rod structure than what has previously been used in the literature, namely one in which the rod directions are normalized to have unit (Euclidean) surface gravity. There is an obvious advantage in adopting this normalization: the condition that there is no conical singularity along a space-like rod requires that the normalized direction of this rod (as a Killing vector field) generates orbits with period 2π. We then show in subsection 3.1.2 that, in order to avoid an orbifold singularity at a so-called turning point where two adjacent space-like rods intersect, their normalized rod directions must generate orbits with period 2π independently. Together, they can be identified as the pair of independent 2π-periodic generators of the $U(1)\times U(1)$ isometry group. Furthermore, they must be related to any other adjacent direction pair of space-like rods by a $GL(2,\mathbb{Z})$ transformation, so that any adjacent direction pair of space-like rods can serve as the pair of independent 2π-periodic generators of the $U(1)\times U(1)$ isometry group. If these conditions are met, the space(-times) are guaranteed to be free of conical and orbifold singularities. The rod structure is a very useful tool to analyze a solution, and it can also be used as characteristics to characterize a solution. We illustrate this by studying the rod structures of some well-known black hole solutions in subsection 3.1.3.

3.1.1 The rod structure

Consider five-dimensional stationary black hole space-times as solutions to the vacuum Einstein equations. We assume, in addition to the Killing vector field corresponding to time flow $V_{(0)} = \frac{\partial}{\partial t}$, the existence of two linearly independent, commuting, space-like Killing vector fields $V_{(1)}$ and $V_{(2)}$, which also commute with $V_{(0)}$.[2] We also assume the following three assumptions hold (here and henceforth in this subsection, we denote $i, j = 0, 1, 2$):

(1) The tensor $V_{(0)}^{[\mu_0} V_{(1)}^{\mu_1} V_{(2)}^{\mu_2} \nabla^\nu V_{(i)}^{\rho]}$ vanishes at at least one point of the space-time for each i.

(2) The tensor $V_{(i)}^\nu R_\nu{}^{[\rho} V_{(0)}^{\mu_0} V_{(1)}^{\mu_1} V_{(2)}^{\mu_2]} = 0$ for each i.

(3) $\det G$ is non-constant in the space-time, where $G_{ij} = g(V_{(i)}, V_{(j)})$ are components of the Gram matrix G, and g is the metric of the space-time.

For the Ricci-flat space-times considered in this thesis, there exists at least one point where some linear combination of $V_{(1)}$ and $V_{(2)}$ vanishes, so conditions (1)–(3) will be trivially satisfied. Such space-times are referred to as stationary and axisymmetric. It was shown in [36] that for such solutions we can find coordinates x^i (with $x^0 = t$), along with ρ and z, such that

$$V_{(i)} = \frac{\partial}{\partial x^i}\,, \tag{3.1}$$

and the metric takes the form

$$\mathrm{d}s^2 = G_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{e}^{2\nu}(\mathrm{d}\rho^2 + \mathrm{d}z^2)\,. \tag{3.2}$$

Here G_{ij} and ν are functions of ρ and z only, and the Gram matrix G is subject

[2]At this point, we do not assume the Killing vector field $V_{(1)}$ or $V_{(2)}$ generates closed orbits. It will be clear below that although $V_{(1)}$ and $V_{(2)}$ together generate a $U(1) \times U(1)$ isometry group, they may not necessarily be the two independent 2π-periodic generators, but instead may be some linear combinations of them, so in general the orbits of $V_{(1)}$ or $V_{(2)}$ are not periodic.

to the constraint

$$\rho = \sqrt{|\det G|}\,. \tag{3.3}$$

The above coordinates (x^i, ρ, z) are usually referred to as Weyl–Papapetrou coordinates. In these coordinates, the vacuum Einstein equations decouple as

$$G^{-1}\left(\partial_\rho^2 + \frac{1}{\rho}\,\partial_\rho + \partial_z^2\right) G = (G^{-1}\partial_\rho G)^2 + (G^{-1}\partial_z G)^2, \tag{3.4}$$

and

$$\begin{aligned} \partial_\rho \nu &= -\frac{1}{2\rho} + \frac{\rho}{8}\,\mathrm{Tr}((G^{-1}\partial_\rho G)^2 - (G^{-1}\partial_z G)^2)\,, \\ \partial_z \nu &= \frac{\rho}{4}\,\mathrm{Tr}(G^{-1}\partial_\rho G G^{-1}\partial_z G)\,. \end{aligned} \tag{3.5}$$

Notice that the integrability of ν in (3.5) is guaranteed by (3.3) and (3.4). Hence, we can always solve the vacuum Einstein equations by first solving for G using (3.3) and (3.4), and subsequently solving for ν using (3.5).

From the condition (3.3), it is clear that the Gram matrix is non-degenerate as long as $\rho > 0$. At $\rho = 0$, it becomes degenerate, so the kernel of $G(\rho = 0, z)$ becomes non-trivial, i.e., $\dim(\ker(G(0,z))) \geq 1$. It was argued in [36] that in order to avoid curvature singularities, it is necessary that $\dim(\ker(G(0,z))) = 1$, except for isolated values of z. When this applies, we label these isolated values as $z_1, z_2, \ldots, z_N$, with $z_1 < z_2 < \cdots < z_N$, and call the corresponding points on the z-axis $(\rho = 0, z = z_i)$ turning points. These turning points divide the z-axis into $N+1$ intervals $(-\infty, z_1]$, $[z_1, z_2]$,..., $[z_{N-1}, z_N]$, $[z_N, \infty)$. These intervals are known as rods, assigned to a given stationary and axisymmetric solution. For clarity of presentation, we label these rods from left to right as rod 1, rod 2, ..., rod $N+1$.

In the interior of a specific rod for $(\rho = 0, z_k < z < z_{k+1})$, the Gram matrix has an exactly one-dimensional kernel. It was further shown in [36] that the kernel is constant along the rod. In other words, we can find a constant nonzero vector

$$v = v^i \frac{\partial}{\partial x^i} = v^i V_{(i)} \,, \tag{3.6}$$

such that

$$G(0, z) v = 0 \,, \tag{3.7}$$

for all $z \in [z_k, z_{k+1}]$. The vector v is assigned to this specific rod and is called its direction. For a given solution, the specification of the rods and the directions associated with them is defined as the (Harmark) rod structure of the solution [36, 37]. Note that in this definition, the direction of a rod is not unique; it can be any nonzero vector in the one-dimensional kernel of the Gram matrix along the rod.

The direction of a rod defined above is a Killing vector field of the space-time, written in the basis consisting of the three linearly independent and mutually commuting Killing vector fields $V_{(i)}$ of the space-time. Without causing confusion, we sometimes also refer to it as the associated Killing vector field of that rod. Along a specific rod $[z_k, z_{k+1}]$, its associated Killing vector field $v = v^i \frac{\partial}{\partial x^i}$ vanishes. It was shown in [36] that near the interior of the rod ($\rho \to 0, z_k < z < z_{k+1}$), we have to leading order $g(v, v) = G_{ij} v^i v^j = \pm a(z) \rho^2$ and $\mathrm{e}^{2\nu} = c^2 a(z)$, where $a(z)$ is a function of z and c a constant. Hence, $\frac{G_{ij} v^i v^j}{\rho^2 \mathrm{e}^{2\nu}}$ tends to a constant in the interior of a rod. If it is negative, positive or zero, the rod is said to be time-like, space-like or light-like, respectively.

For the solutions we are interested in, a rod is either time-like or space-like. A time-like rod represents a Killing horizon. If in addition the Killing vector field corresponding to the flow of time is normalized at infinity, i.e., $g(V_{(0)}, V_{(0)}) = -1$

at infinity, we can choose a particular direction v in the one-dimensional kernel of the Gram matrix for the horizon rod such that $v = (1, \Omega_1, \Omega_2)$, where the constants Ω_1 and Ω_2 are formally defined to be the angular velocities of the horizon, even though in general, the coordinates x^1 and x^2 may not correspond to any axes. The surface gravity on the horizon $\kappa = \left.\sqrt{-\frac{1}{2}v_{\mu;\nu}v^{\mu;\nu}}\right|_H$ (where μ, ν run over all the coordinates) is computed to be $\lim\limits_{\rho\to 0}\sqrt{-\frac{G_{ij}v^iv^j}{\rho^2 \mathrm{e}^{2\nu}}}$. Then we can easily see that the Killing vector field v/κ has unit surface gravity on the horizon.

If the rod $[z_k, z_{k+1}]$ is space-like, it will represent a (two-dimensional) axis for its associated Killing vector field. Consider the orbits generated by the associated Killing vector field near the interior of the rod ($\rho \to 0, z_k < z < z_{k+1}$) along a constant z surface. It is easy to see that there will be a conical singularity unless the orbits generated by the associated Killing vector field $v = \frac{\partial}{\partial\eta}$ (the direction of this rod) are identified with period [36]

$$\Delta\eta = 2\pi \lim_{\rho\to 0}\sqrt{\frac{\rho^2 \mathrm{e}^{2\nu}}{G_{ij}v^iv^j}}\,, \tag{3.8}$$

for $z \in (z_k, z_{k+1})$. We define the Euclidean surface gravity on this rod for its associated Killing vector field v as $\kappa_E = \left.\sqrt{\frac{1}{2}v_{\mu;\nu}v^{\mu;\nu}}\right|_{\mathrm{rod}} = \lim\limits_{\rho\to 0}\sqrt{\frac{G_{ij}v^iv^j}{\rho^2 \mathrm{e}^{2\nu}}}$. Then the Killing vector field v/κ_E will have unit Euclidean surface gravity on the rod, and its orbits should be identified with period 2π in order to avoid a potential conical singularity along the rod.

Thus it is natural to fix the freedom in the direction of a rod by choosing one particular vector in the kernel of the Gram matrix along the interior of the rod, such that it has unit surface gravity for a time-like rod and unit Euclidean surface gravity for a space-like rod.[3] This fixed direction is referred to as the normalized direction of the rod. For a given solution, the specification of the rods and the normalized directions associated with them is referred to as the rod structure of the solution. From now on, the associated Killing vector field of a rod refers only to its normalized direction, and the rod structure of a solution refers only to the stronger version of the rod structure defined here with the rod directions appropriately normalized.

We note that the normalized direction of a rod defined above can differ by a minus sign, but it does not make a difference in the treatment of this thesis. We also note that if the previously defined three linearly independent and mutually commuting Killing vector fields $V_{(i)}$ satisfy conditions (1)–(3), so do the three new Killing vector fields $\tilde{V}_{(i)} = A_{ij}V_{(j)}$ provided the matrix A_{ij} is constant and non-singular. We further take $A_{00} = 1$ and $A_{01} = A_{02} = 0$, so that $\tilde{V}_{(0)} = V_{(0)}$ is also normalized at infinity. Then we can introduce new Weyl–Papapetrou coordinates $(\tilde{x}^i, \tilde{\rho}, \tilde{z})$ such that $\tilde{V}_{(i)} = \frac{\partial}{\partial \tilde{x}^i}$. It can be shown that they are related to the old coordinates (x^i, ρ, z) simply by a linear coordinate transformation

$$x^i = A_{ji}\tilde{x}^j, \qquad \rho = \frac{1}{|\det(A_{ij})|}\,\tilde{\rho}\,, \qquad z = \pm\frac{1}{|\det(A_{ij})|}\,\tilde{z}\,, \tag{3.9}$$

up to harmless translations of x^i and z, all of which are chosen to be zero. Also we always have the freedom to choose $z = \frac{1}{|\det(A_{ij})|}\,\tilde{z}$. So we can clearly see that for a given solution, if $(\rho = 0, z = z_k)$ is its k-th turning point in the old coordinates, then $(\tilde{\rho} = 0, \tilde{z} = |\det(A_{ij})|\, z_k)$ is its k-th turning point in the new coordinates. Furthermore, if $v_k = v^i_k \frac{\partial}{\partial x^i}$ is the normalized direction for its k-th rod in the old coordinates, then $\tilde{v}_k = \tilde{v}^i_k \frac{\partial}{\partial \tilde{x}^i} = v_k$ with $\tilde{v}^i_k = v^j_k (A^{-1})_{ji}$ is the normalized

[3] This normalization of the direction was previously performed, e.g., in [80–82].

direction for its k-th rod in the new coordinates. In other words, the normalized directions, though expressed in the new basis consisting of the three Killing vector fields $\tilde{V}_{(i)}$, are invariant under the above coordinate transformation. Hence, for a physical space-time, different choices of the two space-like Killing vector fields, and so the corresponding Weyl–Papapetrou coordinates, lead to slightly different rod structures. However, in most cases, it is advantageous to make a particular choice of the two space-like Killing vector fields, and thus fully determine the Weyl–Papapetrou coordinates and the corresponding rod structure (up to a minus sign for the directions) for a solution. In the following subsection, the rod structure in standard orientation for different cases is defined by making a particular choice of these two space-like Killing vector fields, so that the rod directions have very simple expressions.

Finally, we point out that the results of this subsection are not necessarily confined to five space-time dimensions. They can readily be extended to any D-dimensional ($D \geq 4$) stationary space-time with $D-3$ linearly independent, mutually commuting, space-like Killing vector fields $V_{(i)}$, $i = 1, \ldots, D-3$.

3.1.2 Regularity conditions

We now focus on the necessary conditions for these black hole space-times to be regular, by which we mean free of conical and orbifold singularities. Let us denote the pair of independent generators of the $U(1) \times U(1)$ isometry group by $\{e_1, e_2\}$, which are assumed to generate the actions of the respective $U(1)$ factors, and are normalized to have period 2π. Then the Killing vector field $\ell = a_1 e_1 + a_2 e_2$ will generate a $U(1)$ isometry subgroup whose orbits are periodic with period 2π if a_1 and a_2 are coprime integers; on the other hand, if a_1 and a_2 are any real numbers other than coprime integers, ℓ will generate an isometry subgroup with orbits that are either non-periodic, or periodic but with a period different from 2π. Suppose the solution has a space-like rod with (normalized) direction v. Recall that to avoid a potential conical singularity, the orbits generated by v must have period 2π in the vicinity of this rod. So it is clear that we must have $v = a_1 e_1 + a_2 e_2$ for some coprime integers a_1 and a_2.[4] Hence, in the basis consisting of (e_1, e_2), the two components of the direction of any space-like rod must be coprime integers.

If two space-like rods $[z_{k-1}, z_k]$ and $[z_k, z_{k+1}]$, with their corresponding directions v_k and v_{k+1} (in the basis (e_1, e_2)), intersect at the turning point $(\rho = 0, z = z_k)$, a further condition should be satisfied [38, 39]:

$$\det(v_k^i, v_{k+1}^j) = \pm 1\,, \tag{3.10}$$

for $i, j = 1, 2$, to avoid a possible orbifold singularity at that point. If this condition is satisfied, the Killing vector fields $\{v_k, v_{k+1}\}$ generate $U(1)$ isometry subgroups with 2π-periodic orbits independently; any Killing vector field that is a linear combination of v_k and v_{k+1} will generate a $U(1)$ isometry subgroup with 2π-periodic orbits if and only if the coefficients are coprime integers. And indeed, since $\{v_k, v_{k+1}\}$ is related to $\{e_1, e_2\}$ by a $GL(2, \mathbb{Z})$ transformation, they can serve as another pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group [38, 39].

[4]We assume that in the basis consisting of $(V_{(0)} = \frac{\partial}{\partial t}, e_1, e_2)$, the direction v of any space-like rod does not have a $V_{(0)}$ component. If this were not the case, periodicity conditions imposed on the orbits of v will impose certain identifications on the time coordinate t.

Hence, for the regular solutions considered in this thesis, we can identify the directions of any two adjacent space-like rods as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group. Without loss of generality, we can take the directions of the left-most pair of adjacent space-like rods to be the pair of independent generators of the isometry group. If these two rod directions are related to the directions of the first and last (semi-infinite) rods by a $GL(2, \mathbb{Z})$ transformation, then the latter two rod directions can serve as the pair of independent generators instead. This is not guaranteed to happen, however. Contrast this to the five-dimensional case in [38, 39], where the asymptotic geometry of the space-times considered is the direct product $M^{1,s} \times T^{5-s-1}$, and the independent generators of the $U(1) \times U(1)$ isometry group are assumed to generate either the standard rotations of the asymptotic Minkowski space-time $M^{1,s}$ or the flat torus T^{5-s-1}. As will be clear below, the solutions considered in this thesis have very different and sometimes more complicated asymptotic geometries, from which the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group cannot be simply identified.

In the case when the first and last (semi-infinite) rods of a particular solution are not parallel, it is useful to introduce Weyl–Papapetrou coordinates defined by taking $\{\tilde{V}_{(1)} = \pm\ell_{N+1}, \tilde{V}_{(2)} = \pm\ell_1\}$, where ℓ_1 and ℓ_{N+1} are the (normalized) directions of the first and last rods respectively.[5] In the event that the first and last rods are parallel, we can introduce Weyl–Papapetrou coordinates defined by taking $\{\tilde{V}_{(1)} = \pm\ell_a, \tilde{V}_{(2)} = \pm\ell_1\}$ instead, where ℓ_a is the direction of the second

[5]Of course, we could have taken $\{\tilde{V}_{(1)} = \pm\ell_1, \tilde{V}_{(2)} = \pm\ell_{N+1}\}$ instead. The particular convention above is used so as to be consistent with the examples considered in section 5.3.

space-like rod from the left such that $\ell_a \neq \pm\ell_1$. In either case, we say that the rod structure has been put in standard orientation. As we shall see in sections 5.3 and 5.4, putting rod structures in standard orientation is a useful way to check if two rod structures are equivalent up to a coordinate transformation, and to compare different rod structures.

On the other hand, in the particular Weyl–Papapetrou coordinates defined by taking $\{V_{(1)}, V_{(2)}\}$ as the pair of independent 2π-periodic generators $\{e_1, e_2\}$ of the $U(1) \times U(1)$ isometry group, the lengths of the rods in the rod structure will be invariants for isometric space-times. Together with the directions of space-like rods, and certain asymptotic quantities, they were used to characterize a solution in [38, 39] for the black hole space-times defined therein. We note that the topology of the event horizon can also be read off from the rod structure; in particular, it is determined solely by the directions of the two space-like rods that are adjacent to the time-like rod representing the horizon.

Even if the solution under consideration is regular, we should point out that Weyl–Papapetrou coordinates are not able to furnish a coordinate chart along the rods. In particular, the metric in these coordinates fails to be analytic at a turning point $(\rho = 0, z = z_k)$ where two space-like rods intersect, and local coordinates need to be constructed. We note that, by identifying the orbits generated by $\{\ell_k, \ell_{k+1}\}$ with period 2π independently, the metric in the vicinity of the turning point for a constant time slice can be brought into the standard form of four-dimensional flat Euclidean space E^4 near the origin [38, 40, 83]:

$$\mathrm{d}s^2 = \mathrm{d}r_1^2 + r_1^2\mathrm{d}\phi_1^2 + \mathrm{d}r_2^2 + r_2^2\mathrm{d}\phi_2^2\,, \tag{3.11}$$

with $r_1, r_2 \geq 0$, and with $\frac{\partial}{\partial\phi_1}$ and $\frac{\partial}{\partial\phi_2}$ identified with ℓ_k and ℓ_{k+1} respectively (so that ϕ_1 and ϕ_2 have period 2π independently). For more detailed aspects of the behaviour of the space-time near the rods and turning points, and the construction of local coordinates at these locations, the reader is referred to [38–40, 83].

It may also be worthwhile to point out a fibre-bundle viewpoint of the space-times considered here, following the approach of [38–40]. Let M be the domain of outer communication of the five-dimensional black hole space-time. Since we assume a $\mathcal{G} = \mathbb{R} \times \mathcal{T}$ isometry group for M, it is interesting to see what the orbit space $\hat{M} = M/\mathcal{G}$ is. For the black hole space-times defined in [38–40], the orbit space $\hat{M}$ is a two-dimensional manifold with boundaries and corners. It turns out that the same thing holds for the space-times considered here.[6] We can further map the orbit space analytically to the upper-half complex plane, and introduce globally defined coordinates (ρ, z) such that $\hat{M} = \{z + i\rho \in \mathbb{C} \mid \rho > 0\}$. These coordinates coincide with the Weyl–Papapetrou coordinates defined above by taking $\{V_{(1)} = e_1, V_{(2)} = e_2\}$, up to a possible translation and reflection of z (we can always appropriately choose them so that they are identical). The boundary $\rho = 0$ of $\hat{M}$ consists of a sequence of line segments and corners (intersections of the line segments). The line segments, which correspond to axes or horizons, coincide with the previously defined rods of the solution; while the corners, which correspond to points where the axes intersect, or to points where axes intersect with

[6] It was assumed in [38], and subsequently proved in [39], that the orbit space $\hat{M}$ does not contain conical singularities due to the possible presence of points with discrete isotropy group. We believe that a similar result will hold for our case under suitable technical assumptions, but a proof of this is beyond the scope of the present thesis.

horizons, coincide with the previously defined turning points. Thus the space-times considered here, with the axes and horizons removed, can be taken as a $\mathcal{G}$-principal fibre bundle over the upper-half complex plane $\hat{M}$, with the projection map naturally sending a point in the space-time to the corresponding point in the orbit space $\hat{M}$.

We remark that the space-times considered in this thesis, as manifolds with $\mathbb{R} \times U(1)^2$ action, are uniquely determined by the rod structure [39, 84, 85]. For any given rod structure satisfying the regularity condition (3.10), the manifold can in principle be constructed from it.

When we remove the black hole, together with the time dimension t, from the space-time (so the Killing vector field $V_{(0)} = \frac{\partial}{\partial t}$ no longer exists), our analysis applies to four-dimensional manifolds I with Euclidean signature, and with an isometry group $\mathcal{T} = U(1) \times U(1)$. This is actually the situation relevant for the gravitational instantons, and will be discussed in chapter 5.

Finally, we note that the above necessary regularity conditions can be generalized to stationary black holes in $D > 5$ space-time dimensions with $\mathbb{R} \times U(1)^{D-3}$ isometry [36, 38, 39]. The orbit space will again be a two-dimensional manifold with boundaries and corners homeomorphic to the upper-half complex plane. However, the directions of any two adjacent space-like rods intersecting at a turning point must now satisfy a new condition instead of (3.10), as shown in [39]. This makes the analysis of the necessary regularity conditions at the turning points of these space-times more involved than the five-dimensional case considered here.

3.1.3 Rod structures of some known black holes

In this subsection we do the rod-structure analysis for some well-known black holes in four and five space-time dimensions, namely, the Kerr black hole, five-dimensional Myers–Perry black hole, and Emparan–Reall Black ring.[7]

Kerr black hole

The above analysis can be readily extended and adjusted to define the rod structure of stationary black holes in four space-time dimensions, in which case the rigidity theorem [11] implies the existence of an axial symmetry $U(1)$. So the isometry group of such black holes is $\mathbb{R} \times U(1)$, as demonstrated by the Kerr black hole (2.1). The rod structure can be defined for this case directly. The difference between the rod structures for four and five-dimensional black holes is that in four dimensions, the direction of a space-like rod can have only one component along the $U(1)$ direction.[8] Hence to avoid any possible conical singularities, all the space-like rods must have the same normalized direction (up to a minus sign), and represent the axes of the $U(1)$ isometry.

In the Kerr black hole, the two factors of the isometry group $\mathbb{R} \times U(1)$ are parameterized by the coordinates t and ϕ respectively. Take the two commuting Killing vector fields as $(V_{(0)} = \frac{\partial}{\partial t}, V_{(1)} = \frac{\partial}{\partial \phi})$. The Weyl–Papapetrou coordinates

[7] The Harmark rod structures of these black holes have been previously studied in [36].

[8] Again, we assume that the direction of a space-like rod does not have a time component. Otherwise, the solution will possess NUT charges.

$(x^0 = t, x^1 = \phi, \rho, z)$ are then related to the coordinates of (2.1) by

$$\rho = \sqrt{r^2 - 2mr + a^2}\,\sin\theta\,, \qquad z = (r - m)\cos\theta\,. \tag{3.12}$$

In these coordinates, the rod structure has two turning points at $(\rho = 0, z = z_1 \equiv -\sqrt{m^2 - a^2})$ or $(r = r_0, \theta = \pi)$, and $(\rho = 0, z = z_2 \equiv \sqrt{m^2 - a^2})$ or $(r = r_0, \theta = 0)$, where $r_0 = m + \sqrt{m^2 - a^2}$. They divide the z-axis into three rods; from left to right they are:

- Rod 1: a semi-infinite space-like rod located at $(\rho = 0, z \leq z_1)$ or $(r \geq r_0, \theta = \pi)$, with direction $\ell_1 = (0, 1)$.
- Rod 2: a finite time-like rod located at $(\rho = 0, z_1 \leq z \leq z_2)$ or $(r = r_0, 0 \leq \theta \leq \pi)$, with direction $\ell_2 = \frac{1}{\kappa}(1, \Omega)$, where Ω and κ are given by $\Omega = \frac{a}{2m(m+\sqrt{m^2-a^2})}$ and $\kappa = \frac{\sqrt{m^2-a^2}}{2m(m+\sqrt{m^2-a^2})}$.
- Rod 3: a semi-infinite space-like rod located at $(\rho = 0, z \geq z_2)$ or $(r \geq r_0, \theta = 0)$, with direction $\ell_3 = (0, 1)$.

The rod structure is depicted in Fig. 3.1.

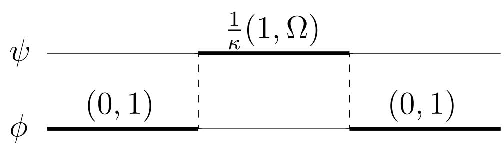

Figure 3.1: The rod structure of the Kerr black hole. The thin lines denote the z axis, and the thick lines denote rods along this axis with directions (vectors) placed above them.

Rod 1 and 3 are space-like and have the same direction. They represent the axes in the black hole space-time. To avoid possible conical singularities along these two rods, the orbits generated by $\frac{\partial}{\partial\phi}$ must be identified with period 2π. This is to say that the coordinate ϕ has period 2π (with other coordinates fixed). Rod 2 is time-like and represents the black hole horizon. Ω is the angular velocity of the black hole, and κ is the surface gravity on the event horizon. From the rod structure, it is clear that the black hole has an event horizon topology S^2, with the north and south poles corresponding to the two turning points, where the Killing vector field $\frac{\partial}{\partial\phi}$ vanishes.

Five-dimensional Myers–Perry black hole

As already mentioned in the section 2.2, the five-dimensional Myers–Perry black hole has an isometry group $\mathbb{R}\times U(1)^2$, where $\mathbb{R}$ corresponds to the flow of time, and the two $U(1)$'s correspond to the two asymptotic axes parameterized by the coordinates ψ and ϕ. We then take the three linearly independent and mutually commuting Killing vector fields to be $(\frac{\partial}{\partial t}, \frac{\partial}{\partial\psi}, \frac{\partial}{\partial\phi})$. The corresponding Weyl–Papapetrou coordinates are then related to coordinates in (2.3) by

$$\begin{aligned} \rho &= \frac{1}{2}\sqrt{(r^2+a^2)(r^2+b^2)-2mr^2}\,\sin 2\theta, \\ z &= \frac{1}{4}\left(2r^2+a^2+b^2-2m\right)\cos 2\theta. \end{aligned} \tag{3.13}$$

In these coordinates, the rod structure has two turning points, at $(\rho=0, z=z_1 \equiv -\frac{1}{4}\sqrt{[2m-(a+b)^2][2m-(a-b)^2]})$ or $(r=r_0, \theta=\frac{\pi}{2})$, and $(\rho=0, z=z_2\equiv -z_1)$ or $(r=r_0, \theta=0)$, where r_0 is the greatest root of Δ. They divide the z-axis into three rods; from left to right they are:

- Rod 1: a semi-infinite space-like rod located at $(\rho = 0, z \leq z_1)$ or $(r \geq r_0, \theta = \frac{\pi}{2})$, with direction $\ell_1 = (0, 0, 1)$.

- Rod 2: a finite time-like rod located at $(\rho = 0, z_1 \leq z \leq z_2)$ or $(r = r_0, 0 \leq \theta \leq \frac{\pi}{2})$, with direction $\ell_2 = \frac{1}{\kappa}(1, \Omega_1, \Omega_2)$, where

$$\Omega_1 = \frac{\sin(\lambda_1 + \lambda_2)}{2\mu(\cos\lambda_1 + \cos\lambda_2)}, \quad \Omega_2 = \frac{\sin(\lambda_1 - \lambda_2)}{2\mu(\cos\lambda_1 + \cos\lambda_2)}, \quad \kappa = \frac{\cos\lambda_1 \cos\lambda_2}{\mu(\cos\lambda_1 + \cos\lambda_2)}. \tag{3.14}$$

  The parameters μ, λ_1 and λ_2 are defined by the relations

$$m = 2\mu^2, \quad a = \mu\left(\sin\lambda_1 + \sin\lambda_2\right), \quad b = \mu(\sin\lambda_1 - \sin\lambda_2). \tag{3.15}$$

- Rod 3: a semi-infinite space-like rod located at $(\rho = 0, z \geq z_2)$ or $(r \geq r_0, \theta = 0)$, with direction $\ell_3 = (0, 1, 0)$.

The rod structure of the five-dimensional Myers–Perry black hole is depicted in Fig. 3.2.

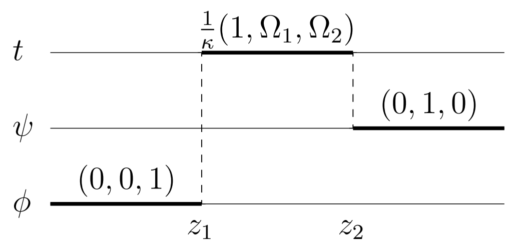

Figure 3.2: The rod structure of the five-dimensional Myers–Perry black hole.

Rod 1 and 3 are space-like and they represent the two asymptotic axes of the five-dimensional Myers–Perry black hole. It is clear that to ensure asymptotic flatness, the orbits of the directions of rod 1 and 3 must be identified with period

2π independently, i.e.,

$$(\psi, \phi) \to (\psi, \phi + 2\pi), \quad (\psi, \phi) \to (\psi + 2\pi, \phi). \tag{3.16}$$

Here and henceforth, we assume implicitly that all the identifications are made for fixed (ρ, z). The direction pair (ℓ_1, ℓ_3) is then identified as the two independent 2π-periodic generators of the $U(1)\times U(1)$ isometry group of this black hole space-time. Rod 2 is time-like and represents the black hole horizon. The black hole is rotating with two angular velocities Ω_1 and Ω_2, with a surface gravity on the horizon κ. From the rod structure, it is clear that the five-dimensional Myers–Perry black hole has an event horizon topology S^3, with the two poles corresponding to the two turning points, where the Killing vector fields $\frac{\partial}{\partial\phi}$ and $\frac{\partial}{\partial\psi}$ vanish respectively.

Emparan–Reall black ring

The Emparan–Reall black ring (2.5) has the same isometry group as the five-dimensional Myers–Perry black hole (2.3). Again we take the three linearly independent and mutually commuting Killing vector fields as $(\frac{\partial}{\partial t}, \frac{\partial}{\partial\psi}, \frac{\partial}{\partial\phi})$. The corresponding Weyl–Papapetrou coordinates are then related to the coordinates in (2.5) by

$$\rho = \frac{2\varkappa^2\sqrt{-G(x)G(y)}}{(x-y)^2}, \quad z = \frac{\varkappa^2(1-xy)(2+cx+cy)}{(x-y)^2}. \tag{3.17}$$

In these coordinates, the rod structure has three turning points, at $(\rho = 0, z = z_1 \equiv -c\varkappa^2)$ or $(x = -1, y = -\frac{1}{c})$, $(\rho = 0, z = z_2 \equiv c\varkappa^2)$ or $(x = 1, y = -\frac{1}{c})$, and $(\rho = 0, z = z_3 \equiv \varkappa^2)$ or $(x = 1, y = -1)$ respectively. They divide the z-axis into four rods; from left to right they are:

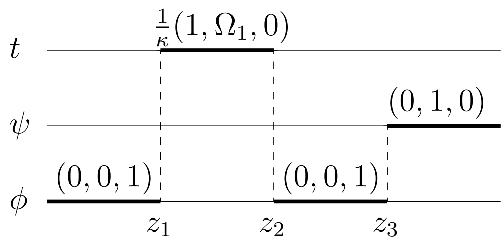

Figure 3.3: The rod structure of the (regular) Emparan–Reall black ring.

- Rod 1: a semi-infinite space-like rod located at $(\rho = 0, z \le z_1)$ or $(x = -1, -\frac{1}{c} \le y \le -1)$, with direction $\ell_1 = (0,0,1)$.

- Rod 2: a finite time-like rod located at $(\rho = 0, z_1 \le z \le z_2)$ or $(-1 \le x \le 1, y = -\frac{1}{c})$, with direction $\ell_2 = \frac{1}{\kappa}(1,\Omega_1,0)$, where Ω_1 and κ are given by

$$\kappa = \frac{(1+c)(1-b)}{k(1-c)\sqrt{8cb(1+b)}}, \quad \Omega_1 = \frac{1}{k(1-c)}\sqrt{\frac{(b-c)(1-b)}{2b(1+b)}}. \tag{3.18}$$

- Rod 3: a finite space-like rod located at $(\rho = 0, z_2 \le z \le z_3)$ or $(x = 1, -\frac{1}{c} \le y \le -1)$, with direction $\ell_3 = (0, 0, \frac{1-c}{1+c}\sqrt{\frac{1+b}{1-b}})$.

- Rod 4: a semi-infinite space-like rod located at $(\rho = 0, z \ge z_3)$ or $(-1 \le x \le 1, y = -1))$, with direction $\ell_4 = (0,1,0)$.

The rod structure of the (regular) Emparan–Reall black ring is depicted in Fig. 3.3.

Rod 3 and 4 are space-like, and they intersect at the third turning point $(\rho = 0, z = z_3)$. According to the analysis in the previous subsections, the orbits generated by the directions of rod 3 and 4 must be identified with period 2π independently, i.e.,

$$(\psi,\phi) \to (\psi,\phi+2\pi), \quad (\psi,\phi) \to (\psi+2\pi,\phi). \tag{3.19}$$

The direction pair (ℓ_3, ℓ_4) is then identified as the two independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group of the Emparan–Reall black ring. Rod 3 represents an inner axis bounded by the S^1 of the event horizon. To avoid a conical singularity along rod 1, we should require $\ell_1 = \ell_3$, and then we recover the balance condition of the black ring $b = \frac{2c}{1+c^2}$. In this case it can be checked that the identification (3.19) ensures the asymptotic flatness of the space-time, with rod 1 and 4 representing the two asymptotic axes. Rod 2 is time-like and represents the black hole horizon. The black hole is rotating along the S^1 direction parameterized by ψ with angular velocity Ω_1. The surface gravity on the horizon is κ. From the rod structure, it is clear that the Emparan–Reall black ring has an event horizon topology $S^1 \times S^2$, with the azimuthal angle of the S^2 parameterized by ϕ.

3.2 Solution-generating techniques

Historically Emparan and Reall found their black ring solution by educated guesswork [19]. However, systematic ways to derive this and new solutions in higher dimensions are desired, especially in the light of the anticipation that black holes in higher dimensions would reveal very rich phase structures. Several solution-generating techniques have been developed along the way to achieve this goal.

For five-dimensional vacuum gravity with isometry group $\mathbb{R} \times U(1)^2$, the Einstein equations are completely integrable and reduce to a two-dimensional non-linear sigma model as shown in the previous section. For such systems, a Bäcklund transformation, which was originally used in four-dimensional gravity [86], can be applied on some simple solutions known as seed solutions, to generate new solutions with angular momentum along a single rotational axis [87–89]. Another method that has been developed is the inverse scattering method (ISM) [44–46], by Belinsky and Zakharov. A clever application of this method to generate five-dimensional solutions, was employed by Pomeransky [47]. For the system described above, it is also possible to view it as a three-dimensional non-linear sigma model instead, which is known to possess a hidden $SL(3,\mathbb{R})$ symmetry [90]. By applying $SL(3,\mathbb{R})$ transformations on such a system, many new solutions have also been obtained [62, 91, 92]. Among these solution-generating techniques, the inverse scattering method is the most powerful and successful one. In what follows in this section, we will first briefly review this method, following the approach used by Pomeransky in [47] (see also the review [20]), and then sketch how to use this method to generate some well-known black hole solutions.

3.2.1 Inverse scattering method

In fact, the inverse scattering method can be applied to generate black hole solutions in D dimensions with an isometry group $\mathbb{R} \times U(1)^{D-3}$. As mentioned in the previous section, for such solutions, we can find Weyl–Papapetrou coordinates such that the metric takes the form as (3.2). The Einstein equations then decouple as two groups of equations, namely (3.4) and (3.5). If we define two $(D-2)\times(D-2)$ matrices U and V as follows,

$$U = \rho(\partial_\rho G)G^{-1}, \quad V = \rho(\partial_z G)G^{-1}, \tag{3.20}$$

then Eq. (3.4) becomes

$$\partial_\rho U + \partial_z V = 0. \tag{3.21}$$

It is known that the above system is completely integrable [44, 45]. It admits a Lax pair of linear differential equations:

$$D_\rho \psi = \frac{\rho U + \lambda V}{\lambda^2 + \rho^2}\psi, \quad D_z \psi = \frac{\rho V - \lambda U}{\lambda^2 + \rho^2}\psi. \tag{3.22}$$

Here $\psi(\rho, z, \lambda)$ is a complex matrix and λ is a complex parameter called "spectral parameter". The differential operators D_ρ and D_z are defined as

$$D_\rho = \partial_\rho + \frac{2\lambda\rho}{\lambda^2 + \rho^2}\partial_\lambda, \quad D_z = \partial_z - \frac{2\lambda^2}{\lambda^2 + \rho^2}\partial_\lambda. \tag{3.23}$$

It is easy to check that the system (3.22) is compatible if and only if there is a metric matrix G satisfying (3.20) and (3.21). Notice that we can extract G from ψ by $G = \psi(\rho, z, \lambda = 0)$.

Since the Lax pair of differential equations (3.22) is linear, it allows us to generate an infinite number of solutions starting from known ones following a purely algebraic procedure. We can thus construct new solutions by "dressing" a known "seed" solution. From a seed metric G_0, we can compute the corresponding matrices U_0 and V_0, and then solve ψ_0 from (3.22). The dressing procedure involves a new matrix χ, which is the key quantity to be determined, and the new solution is written in the form

$$\psi = \chi \psi_0. \tag{3.24}$$

Now the task is to solve the matrix χ, which admits the most interesting solutions called solitonic ones in terms of simple poles

$$\chi = 1 + \sum_{k=1}^{n} \frac{R_k}{\lambda - \mu_k}, \tag{3.25}$$

where the residue matrices R_k and the "pole-position" functions μ_k depend only on the coordinates ρ and z. Each pole corresponds to a solition, so the number of poles is the number of solitons. It is straightforward to determine the pole-positions as $\mu_k = \pm\sqrt{\rho^2 + (z - w_k)^2} - (z - w_k)$, where w_k are in general complex constants. We shall refer to the cases of signs plus and minus before the square root in this expression as soliton and anti-soliton respectively.

The solutions for the matrices R_k can be constructed by first introducing a set of vectors (column matrices) $m^{(k)}$ using the seed as

$$m^{(k)} = [\psi_0^{-1}(\mu_k)]^T m_0^{(k)}, \tag{3.26}$$

where $m_0^{(k)}$ are constant vectors, with components known as BZ parameters. These parameters are the crucial new parameters introduced in the resulting solution. Define a symmetric matrix Γ_{kl} and its inverse D^{kl} as

$$\Gamma_{kl} = \frac{[m^{(k)}]^T G_0 m^{(l)}}{\rho^2 + \mu_k \mu_l}, \quad D^{km}\Gamma_{ml} = \delta_l^k. \tag{3.27}$$

Then the matrices R_k can be expressed as

$$R_k = \sum_l \mu_l^{-1} D^{kl} G_0 m^{(l)} (m^{(k)})^T. \tag{3.28}$$

We can see the solutions thus generated are characterized by the BZ parameters contained in the vectors $m_0^{(k)}$ and the positions of solitons w_k contained in μ_k. In terms of these parameters, the new metric we obtain is

$$\begin{aligned} G &= G_0 + \sum_{k,l} \frac{1}{\mu_l(\lambda - \mu_k)} D^{kl} G_0 m^{(l)} (m^{(k)})^T \psi_0 \Big|_{\lambda=0}, \\ &= G_0 - \sum_{k,l} \frac{D^{kl}}{\mu_k \mu_l} G_0 m^{(l)} (G_0 m^{(k)})^T. \end{aligned} \tag{3.29}$$

In the special case when the seed metric is diagonal, and each BZ vector $m_{0a}^{(k)}$ has only one non-zero component, the resulting metric is also diagonal. It is obtained from G_0 simply by multiplying its diagonal elements g_{0aa} corresponding to the non-zero components of $m_{0a}^{(k)}$ by $-\rho^2/\mu_k^2$.

The metric thus obtained has a determinant $\det G = (-\rho^2)^n (\prod_{k=1}^n \mu_k^{-2}) \det G_0$. This, however, does not obey the condition (3.3), which must be satisfied for a physical solution. A clever and very practical way out of this problem was proposed by Pomeransky [47]. The key idea is the observation that the above determinant is independent of the BZ parameters. One may then start with a seed solution satisfying condition (3.3), remove a number of solitons with certain BZ parameters, and then readd these same solitons, but now with different BZ parameters. Here removing a soliton/anti-soliton just refers to the operation of adding an anti-soliton/soliton at the same position. Notice that the multiplication of a soliton and an anti-soliton at the same position is nothing but $-\rho^2$. The metric thus obtained then automatically satisfies the condition (3.3).

In practice, we can always start from a diagonal, hence static solution $(G_0, e^{2\nu_0})$. Such seed solutions, called generalized Weyl solutions, have been studied in detail by Emparan and Reall in [35]. These solutions are completely determined by and can be simply read off from their rod structures. Notice that for a diagonal seed G, it can always be taken as functions of ρ and certain number of solitons μ_i. For such solutions, it is direct to compute their corresponding ψ_0 matrix. ψ_0 is obtained by doing the following replacement of the G matrix: $\mu_i \to \mu_i - \lambda$, $\rho^2 \to \rho^2 - 2z\lambda - \lambda^2$ or $\frac{\rho^2}{\mu_i} \to \frac{\rho^2}{\mu_i} + \lambda$.

We then remove at rod junctions some solitons and anti-solitons with "trivial" vectors $m_0^{(k)} = \xi_{(a)}$, each aligned with the directions of one of the rods that intersect at the corresponding rod junctions. Notice that the only non-vanishing components of these trivial vectors can be taken to be 1, since it can be shown that we have the freedom to rescale each of the BZ vectors by a constant. We then add the solitons and anti-solitons back, with more general vectors $\tilde{m}_0^{(k)}$. Each $\tilde{m}_0^{(k)}$ differs from $m_0^{(k)}$ by one more non-vanishing component along the direction of the other rod that intersect at the corresponding rod junction. These new components are the new parameters introduced. Of course $\tilde{m}_0^{(k)}$ can have all components non-vanishing to generate new solutions, which in general, however, possess naked singularities. The conformal factor $e^{2\nu}$ of the resulting solution can be computed to be

$$e^{2\nu} = e^{2\nu_0} \frac{\det \Gamma}{\det \Gamma_0}, \tag{3.30}$$

where Γ_0 and Γ are obtained from (3.27) in the above two ISM operations of removing and readding solitons and anti-solitons respectively.

The method described above is very powerful, and has been very successfully applied to generate solutions in four and five dimensions. Various new solutions, including those with disconnected horizons, have been constructed. In particular, it has been successfully used to generalize a static solution by adding angular momenta to it. It has been the general feeling that all stationary and axisymmetric solutions can be generated using this method [35]. However, we note that in the above method, we have various choices of the seed solutions and the BZ vectors, among many others, which may result in equivalent or related solutions. There is no general rule that can be used to ensure that the final solutions have no pathologies; in general, we have to examine them case by case. Moreover, these final solutions are always local metrics. From the construction itself, we can not tell whether they can be interpreted as global space-times.

3.2.2 ISM construction of some known black holes

Various solutions have been generated by using the inverse scattering method reviewed above. In this subsection, we sketch how to use this method to construct some well-known black hole solutions, including the Kerr black hole, five-dimensional Myers–Perry black hole, and Emparan–Reall black ring. These constructions have been reviewed in [20]. As we already mentioned, these constructions may not be unique. Here we try to use the simplest and most direct ones. The reader is referred to [20] for more relevant aspects of constructions of these black holes, and to [35] to see how to read off the seed solutions from their rod structures.

Kerr black hole

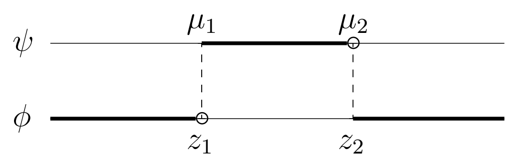

Figure 3.4: The rod structure of the seed for the Kerr black hole. The thin lines denote the z axis and the thick lines denote rods along this axis. The direction of a rod has a single non-vanishing component along the coordinate that labels the z axis where the rod is placed. Small circles represent operations of removing solitons from the seed, with BZ vector aligned with the direction of the corresponding rod.

To generate the Kerr black hole solution, we use its static limit, the Schwarzschild solution, as seed. The rod structure of the Schwarzschild solution is shown in Fig. 3.4. We perform the following BZ operations:

1. remove a soliton at each of z_1 and z_2, with BZ vectors $(0,1)$ and $(1,0)$ respectively;

2. add the same soliton back at each of z_1 and z_2, with BZ vectors $(C_1,1)$ and $(1,C_2)$ respectively.

We call the solution after performing operation 1 as modified seed solution $\tilde{G}_0$. The seed and modified seed solutions are:

$$\begin{aligned} G_0 &= \begin{bmatrix} -\frac{\mu_1}{\mu_2} & 0 \\ 0 & \frac{\rho^2}{\mu_1}\mu_2 \end{bmatrix}, \qquad \tilde{G}_0 = G_0 \begin{bmatrix} -\frac{\mu_2^2}{\rho^2} & 0 \\ 0 & -\frac{\mu_1^2}{\rho^2} \end{bmatrix} = \begin{bmatrix} \frac{\mu_1\mu_2}{\rho^2} & 0 \\ 0 & -\mu_1\mu_2 \end{bmatrix}. \\ e^{2\nu_0} &= \frac{\mu_2 R_{12}^2}{\mu_1 R_{11} R_{22}}, \end{aligned} \tag{3.31}$$

where the functions R_{ij} are defined as $R_{ij} = \rho^2 + \mu_i\mu_j$.

The resulting solution is in fact the Kerr-NUT solution, with angular momentum and NUT charge associated with the two new introduced parameters C_1 and C_2. We can identify the NUT charge and set it to zero. After appropriate coordinate transformations, we obtain the Kerr solution (2.1).

Five-dimensional Myers–Perry black hole

To generate the five-dimensional double-rotating Myers–Perry black hole, we use the five-dimensional Schwarzschild black hole as seed, whose rod structure is shown

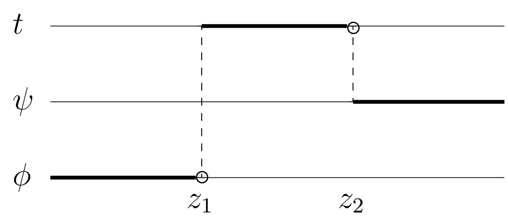

Figure 3.5: The rod structure of the seed for the five-dimensional Myers–Perry black hole.

in Fig. 3.5. We perform the following BZ operations:

1. remove a soliton at each of z_1 and z_2, with BZ vectors $(0, 0, 1)$ and $(1, 0, 0)$ respectively;
2. add the same soliton back at each of z_1 and z_2 , with BZ vectors $(C_1, 0, 1)$ and $(1, C_2, 0)$ respectively.

The seed and modified seed solutions are:

$$
\begin{aligned}
G_0 &= \begin{bmatrix} -\frac{\mu_1}{\mu_2} & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \frac{\rho^2}{\mu_1} \end{bmatrix}, \\
\tilde{G}_0 &= G_0 \begin{bmatrix} -\frac{\mu_2^2}{\rho^2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{\mu_1^2}{\rho^2} \end{bmatrix} = \begin{bmatrix} \frac{\mu_1\mu_2}{\rho^2} & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & -\mu_1 \end{bmatrix}. \\
e^{2\nu_0} &= \frac{\mu_2 R_{12}}{R_{11} R_{22}}\,. \qquad (3.32)
\end{aligned}
$$

The two new parameters C_1 and C_2 introduced are associated with the two angular momenta. After appropriate coordinate transformations [47], we can bring the final solution to the well-known form of the five-dimensional Myers–Perry black hole (2.3).

Emparan–Reall black ring

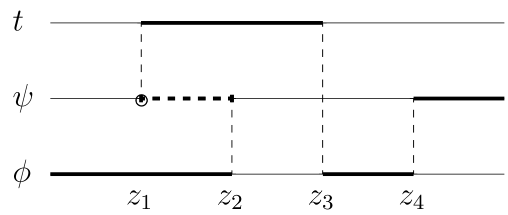

Figure 3.6: The rod structure of the seed for the Emparan–Reall black ring. The dashed horizontal line indicates a rod with negative density $-\frac{1}{2}$.

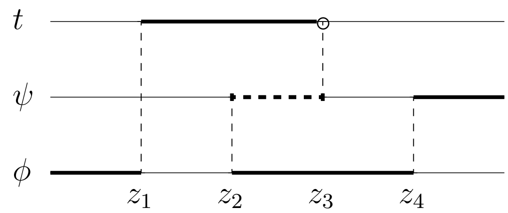

Figure 3.7: The rod structure of an alternative seed for the Emparan–Reall black ring.

To generate the Emparan–Reall black ring, we use a deliberately chosen seed whose rod structure is shown in Fig. 3.6. The seed has a rod with negative density $-\frac{1}{2}$, and thus possesses naked singularities [35]. There are four turning points $z_{1\ldots4}$, which divide the z axis to five rods. We perform the following BZ operations:

1. remove a soliton at z_1, with BZ vector $(0, 1, 0)$;
2. add the same soliton back at z_1 , with BZ vector $(C_1, 1, 0)$.

The seed and modified seed solutions are:

$$
\begin{aligned}
G_0 &= \begin{bmatrix} -\frac{\mu_1}{\mu_3} & 0 & 0 \\ 0 & \frac{\mu_2}{\mu_1}\mu_4 & 0 \\ 0 & 0 & \frac{\rho^2}{\mu_2}\frac{\mu_3}{\mu_4} \end{bmatrix}, \\
\tilde{G}_0 &= G_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{\mu_1^2}{\rho^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\mu_1}{\mu_3} & 0 & 0 \\ 0 & -\frac{\mu_1\,\mu_2\,\mu_4}{\rho^2} & 0 \\ 0 & 0 & \frac{\rho^2\mu_3}{\mu_2\mu_4} \end{bmatrix}. \\
e^{2\nu_0} &= \frac{\mu_2\mu_4 R_{13}R_{12}R_{14}R_{23}R_{34}}{\mu_1 R_{24}^2 R_{11}R_{22}R_{33}R_{44}}. \qquad (3.33)
\end{aligned}
$$

Using this seed, the resulting solution automatically has standard orientation, i.e., we have $\ell_1 = (0, 0, 1)$ and $\ell_5 = (0, 1, 0)$. Now rod 2 is parallel with rod 1, so we must impose that $\ell_2 = (0, 0, \pm 1)$,[9] which fixes the value of the parameter C_1. By imposing this condition, rod 1 and rod 2 in fact now join up to a single rod, so now the first turning point z_1 is effectively eliminated. This is confirmed by casting the final solution to C-metric coordinates [50], which in general, is only possible for a solution with three turning points. We then at last obtain the Emparan–Reall black ring solution (2.5).

The rod with negative density in the seed solution can also be placed at $z_2 < z < z_3$, see Fig. 3.7. The resulting solution differs from the solution above in the preceding

[9] If such a condition is not satisfied, the solution will possess a conical singularity at either rod 1 or rod 2, as well as a naked singularity at the first turning point $z = z_1$.

paragraph by a linear transformation of the G matrix. One may also wonder whether the Emparan–Reall black ring solution can be generated by using the static black ring as a seed. The answer is no; all solutions generated in this way are singular ones.

From this example, we see that a seed solution may not necessarily be regular. The above seed solutions have both naked singularities and conical singularities. Even so, a completely regular solution, the regular Emparan–Reall black ring can be generated. On the other hand, an obvious result is that a regular seed does not guarantee a regular final solution.

Chapter 4

Black lenses

It has recently been shown that a stationary, asymptotically flat vacuum black hole in five space-time dimensions with $\mathbb{R} \times U(1)^2$ isometry must have an event horizon with either a spherical, ring or lens-space topology. In this chapter, we study the third possibility, a so-called black lens with $L(n,1)$ horizon topology. Using the inverse scattering method, we construct a black lens solution with the simplest possible rod structure, and possessing a single asymptotic angular momentum. Its properties are then analyzed; in particular, it is shown that there must either be a conical singularity or a naked curvature singularity present in the space-time.

4.1 Introduction

We have already seen that black holes in five dimensions have much richer phase structures than their four-dimensional counterparts. Hollands and Yazadjiev [38] have recently considered how a uniqueness result might be proved for black holes in five dimensions. They showed that stationary, asymptotically flat vacuum black holes with $\mathbb{R} \times U(1)^2$ isometry are uniquely determined by their mass, angular momentum, and the rod structure. In particular, it is the rod structure that determines the topology of the event horizon, and it was shown that there are only three possibilities: a 3-sphere S^3, a ring $S^2 \times S^1$, or a lens space $L(p, q)$, consistent with the results of [26–28]. The first two cases are just the five-dimensional Myers–Perry black hole and the Emparan–Reall black ring, and it is the purpose here to investigate the third possibility—a so-called black lens.[1]

Supposing that such a black lens solution exists, Hollands and Yazadjiev [38] showed that the simplest rod structure it could take is the one depicted in Fig. 4.1. As can be seen, there is a finite time-like rod (rod 2) which represents the event horizon, and two semi-infinite space-like rods (rod 1 and 4) which are the usual asymptotic axes. Asymptotic flatness requires that the directions of the two semi-infinite rods (ℓ_1, ℓ_4) should be identified as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry. The new feature lies in the finite space-like rod (rod 3), whose direction ℓ_3 has components in both $\frac{\partial}{\partial \psi}$ and $\frac{\partial}{\partial \phi}$. We know from the regularity conditions analyzed in subsection 3.1.2 that (ℓ_3, ℓ_4) can also serve as

[1] Black holes with lens-space horizon topology have previously been considered in the literature, but these have been in non-asymptotically flat spaces. Examples include (multi-)Taub-NUT space [93, 94].

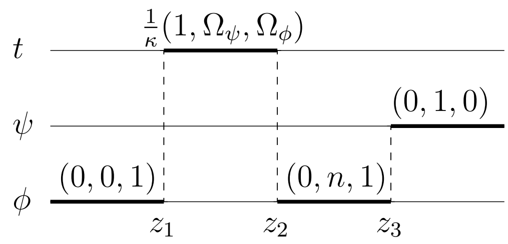

Figure 4.1: The rod structure of the rotating black lens solution.

the pair of independent 2π-periodic generators, and they must be related to (ℓ_1, ℓ_4) by a $GL(2, \mathbb{Z})$ transformation. This then implies $\ell_3 = (0, n, 1)$, for some integer n. Two special cases can immediately be seen: Firstly, if it is $\ell_3 = (0, 0, 1)$, then the horizon topology is $L(0, 1) \cong S^2 \times S^1$ which corresponds to the Emparan–Reall black ring. Secondly, as a degenerate limit, if it is $\ell_3 = (0, 1, 0)$, then the horizon topology is $L(1, 0) \cong S^3$ which corresponds to the five-dimensional Myers–Perry black hole.

A third special case of particular interest is: if $n = 1$, then the horizon topology is $L(1, 1)$, which is again S^3 since $L(1, 1) \cong S^3$; thus, if the corresponding solution exists, it would describe a new type of asymptotically flat black hole with spherical horizon topology S^3 in five dimensions, different from the usual Myers–Perry one. We note that the existence of such type of black hole is not ruled out by the uniqueness theorem of five-dimensional stationary black holes with spherical horizon topology in asymptotically flat space-times proved by Morisawa and Ida [95], since in [95], it was essentially assumed that there exists only two turning points. In our context when $n = 1$, the rod structure has three turning points, so the corresponding solution will not be in the class considered in [95].

Recently, Evslin [96] made a first attempt towards constructing an explicit black lens solution. However, he did not consider the rod structure in Fig. 4.1, but rather one with a second finite space-like rod inserted to the left of the time-like rod, with direction $(0, 1, -n)$. The presence of this extra rod means the event horizon would have the more restrictive lens-space topology $L(n^2 + 1, 1)$. Evslin managed to construct a static metric associated to this rod structure; furthermore, he found that while conical and orbifold singularities could be eliminated from the space-time, there exist spherical naked curvature singularities surrounding each of the two junctions where the space-like rods meet. He went on to conjecture that these singularities could somehow be resolved, possibly by making the black lens rotate.

In this chapter, we shall revisit the simpler rod structure of Fig. 4.1, and construct an asymptotically flat black lens solution associated to it. This is done using the inverse scattering method, and indeed we are able to derive a solution that possesses an asymptotic angular momentum in the ψ direction. In the static limit, this solution was actually found independently in [62] and [82], although it was not interpreted as an asymptotically flat black lens in either paper. We find that even with rotation present, the black lens either has to have a conical singularity along the finite space-like rod, or a naked singularity with spherical topology surrounding the junction z_3 similar to what Evslin found in his static solution. Of these two possibilities, we actually prefer the former interpretation for reasons that would be explained below.

For clarity of presentation, the static and rotating cases will be discussed separately in this chapter. We begin in section 4.2 by presenting the static black lens solution. Its properties are then analysed with particular attention paid to the global structure of the space-time, including the possible existence of conical and curvature singularities outside the horizon. The black lens with a single angular momentum is then presented and analyzed in section 4.3, emphasizing mainly the differences introduced by the rotation. The background and certain black-hole limits of the solution are studied in section 4.4. Section 4.5 concludes this chapter with a discussion of our results.

4.2 Static black lens

The space-time solution describing a static black lens was first derived in [62], although it was not interpreted as such, being used as a stepping-stone to construct a black ring on Taub-NUT space. This solution was obtained using the inverse scattering method [44–46], starting from the static black ring solution [35] as the seed. We sketch the ISM construction of the static black lens in appendix A.

While this solution was originally derived in Weyl–Papapetrou coordinates, it turns out to have a simpler form in C-metric type coordinates (3.17) familiar from the black ring case [36, 71]. In the C-metric type coordinates, the metric reads[2]

$$\begin{aligned} \mathrm{d}s^2 = -\frac{1+cy}{1+cx}\,\mathrm{d}t^2 \quad &+ \quad \frac{2\varkappa^2(1+cx)}{(1-a^2)(x-y)^2H(x,y)}\left\{\frac{H(x,y)^2}{1-c}\left(\frac{\mathrm{d}x^2}{G(x)}-\frac{\mathrm{d}y^2}{G(y)}\right)\right.\\ &+ \quad (1-x^2)\left[(1-c-a^2(1+cy))\mathrm{d}\phi - ac(1+y)\mathrm{d}\psi\right]^2\\ &- \quad \left.(1-y^2)\left[(1-c-a^2(1+cx))\mathrm{d}\psi - ac(1+x)\mathrm{d}\phi\right]^2\right\}, \end{aligned}$$

[2]Some minor notational differences compared to those used in the literature: coordinates (x, y) are used instead of (u, v) in [36], while $a \to -a$ and $\phi \leftrightarrow \psi$ compared to [62].

$$(4.1)$$

where the functions G and H are defined as

$$G(x) = (1-x^2)(1+cx)\,, \qquad H(x,y) = (1-c)^2 - a^2(1+cx)(1+cy)\,. \tag{4.2}$$

As in the black ring case, $\varkappa$ is a scale parameter while the parameter c takes the range $0 < c < 1$. The new parameter a takes the range $-1 < a < 1$ to ensure the correct space-time signature. The coordinates t, ψ and ϕ take the ranges $-\infty < t < \infty$, $-1 \leq x \leq 1$, $-1/c \leq y \leq -1$. Note that the metric is invariant under the action $a \to -a$ and either $\psi \to -\psi$ or $\phi \to -\phi$.

In these coordinates, the rod structure has three turning points, at $(\rho = 0, z = z_1 \equiv -c\varkappa^2)$ or $(x = -1, y = -\frac{1}{c})$, $(\rho = 0, z = z_2 \equiv c\varkappa^2)$ or $(x = 1, y = -\frac{1}{c})$, and $(\rho = 0, z = z_3 \equiv \varkappa^2)$ or $(x = 1, y = -1)$ respectively. They divide the z-axis into four rods; from left to right they are:

- Rod 1: a semi-infinite space-like rod located at $(\rho = 0, z \leq z_1)$ or $(x = -1, -\frac{1}{c} \leq y \leq -1)$, with direction $\ell_1 = (0, 0, 1)$.
- Rod 2: a finite time-like rod located at $(\rho = 0, z_1 \leq z \leq z_2)$ or $(-1 \leq x \leq 1, y = -\frac{1}{c})$, with direction $\ell_2 = \frac{1}{\kappa}(1, 0, 0)$, where $\kappa = \frac{1}{4\varkappa c}\sqrt{2(1-a^2)(1+c)}$.
- Rod 3: a finite space-like rod located at $(\rho = 0, z_2 \leq z \leq z_3)$ or $(x = 1, -\frac{1}{c} \leq y \leq -1)$, with direction $\ell_3 = (0, \frac{n}{m}, \frac{1}{m})$, where
$$n = \frac{2ac}{1-c-a^2(1+c)}, \qquad m = \frac{(1-a^2)\sqrt{(1-c^2)}}{1-c-a^2(1+c)}. \tag{4.3}$$
- Rod 4: a semi-infinite space-like rod located at $(\rho = 0, z \geq z_3)$ or $(-1 \leq x \leq 1, y = -1)$, with direction $\ell_4 = (0, 1, 0)$.

Asymptotic infinity is located at $(x, y) \to (-1, -1)$, and it is easy to see from the rod structure that to ensure asymptotic flatness, ψ and ϕ must have period 2π independently, so that (ℓ_1, ℓ_4) can be identified as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group.

Some special cases can immediately be read off from the above result. One is when $a = 0$, in which rod 1 is parallel to rod 3. In this event, we recover the static black ring with $S^2 \times S^1$ event-horizon topology (with a conical singularity). The other is when $a = \pm\sqrt{(1-c)/(1+c)}$, in which rod 3 is parallel to (indeed, joined up to) rod 4. In this event, we recover the usual five-dimensional Schwarzschild black hole with S^3 horizon topology (see subsection 4.4.2). The background limit is recovered when $c \to 0$ (see subsection 4.4.1).

To obtain a black lens with horizon topology $L(n, 1)$, we require that n is an integer. This can be solved in terms of a as

$$a = \frac{\pm\sqrt{c^2 + n^2(1-c^2)} - c}{n(1+c)} \,. \tag{4.4}$$

The geometry of the event horizon in this case will be studied in detail later in this section. For simplicity we consider here the case $n > 0$; the case $n < 0$ can be similarly analyzed. It can be shown that the above solution with positive sign satisfies $0 < a < \sqrt{(1-c)/(1+c)}$; we call this Range I. On the other hand, the solution with negative sign satisfies $-1 < a < -\sqrt{(1-c)/(1+c)}$; we call this Range II. It is instructive to plot n against a (for fixed c) to see these two ranges, as in Fig. 4.2. Note that in Range I, n takes integer values in the interval $(0, \infty)$; while in Range II, n takes integer values in the interval $(1, \infty)$.

We now turn to a study of possible conical singularities in the space-time. Recall

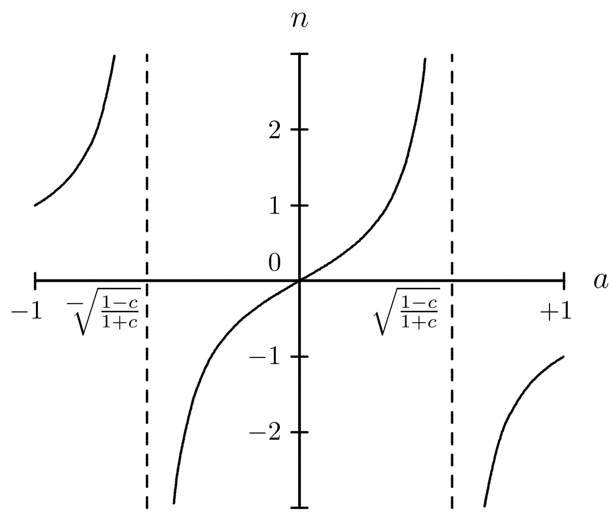

Figure 4.2: Graph of n against a, for fixed c.

that we have identified (ℓ_1, ℓ_4) as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group. This implies that, for integer n, the coordinate associated with the Killing vector field $\partial/\partial\tilde{\psi} \equiv n\partial/\partial\psi + \partial/\partial\phi$ that vanishes along the $x = 1$ axis has period $\Delta\tilde{\psi} = 2\pi$. Hence to ensure regularity (see subsection 3.1.2) we require $m = \pm 1$; otherwise, there will be a conical singularity along rod 3. Solving the condition $m = \pm 1$ gives

$$a = \pm\sqrt[4]{(1-c)/(1+c)}\,. \tag{4.5}$$

The positive solution does not lie in Range I, so the conical singularity cannot be eliminated for any a in this range; indeed, it can be shown that $m^2 > n^2$. However, the negative solution of (4.5) lies in Range II, so the conical singularity can be eliminated for this particular value of a. Imposing the two conditions (1) n is a positive integer and (2) $m = \pm 1$ simultaneously, we see that it corresponds to the solution

$$a = \frac{\sqrt{n^2-4}-n}{2}\,, \qquad c = \frac{1-a^4}{1+a^4}\,, \tag{4.6}$$

and is only applicable for $n \geq 3$. (The $n = 2$ case is excluded, as it implies $c = 0$, which means that there is no longer a black lens present in the space-time.) The allowed range of a for this solution is $-1 < a < 0$.

While it may seem from the preceding result that Range II is the more appropriate range to consider, all solutions in this range unfortunately suffer from the following pathology: It can be checked that the value of $H(x, y)$ is zero on a closed surface which separates the point $(x, y) = (1, -1)$ from the rest of the space-time, including the horizon and asymptotic infinity. Since the curvature invariant $R_{abcd}R^{abcd} \sim H(x, y)^{-6}$, this surface is a singular one; moreover, it is nakedly singular since it is not enclosed by the event horizon. Since this surface intersects the $y = -1$ and $x = 1$ axes, it has an $L(1, n) \cong S^3$ spherical topology. On the other hand, it can be checked that for Range I, $H(x, y)$ is positive everywhere in the space-time outside the event horizon, so this naked singularity does not exist. For either range, if we extend the coordinate range below the horizon $y < -1/c$, we would also find a curvature singularity at $y \to -\infty$.

At this stage, let us explicitly examine the horizon geometry of our solution to confirm the above interpretation that it has an $L(n, 1)$ lens-space topology. Our analysis will follow that of [82]. From (4.1), its metric on a constant time slice is given by

$$
\begin{aligned}
\mathrm{d}s_{\mathrm{H}}^2 \;=\; & \frac{2\varkappa^2}{(1-a^2)(1-c)(1+cx)}\Bigg\{c^2(1-c)^2\frac{\mathrm{d}x^2}{G(x)} - 2ac(1+x)(1+cx)\times \\
& \big[1-c-a^2(1+c)\big]\mathrm{d}\psi\mathrm{d}\phi + \Big[(1+c)\left(1-c-a^2(1+cx)\right)^2 + a^2c^2(1-c) \\
& \times(1-x^2)\Big]\mathrm{d}\psi^2 + c^2(1+x)\left[(1-c)(1-x) + a^2(1+c)(1+x)\right]\mathrm{d}\phi^2\Bigg\}.
\end{aligned}
\tag{4.7}
$$

We now introduce two Killing vector fields $\frac{\partial}{\partial\tilde{\phi}}$ and $\frac{\partial}{\partial\tilde{\psi}}$, defined by

$$\frac{\partial}{\partial\tilde{\phi}} = \ell_1\,, \qquad \frac{\partial}{\partial\tilde{\psi}} = \ell_1 + n\ell_4\,, \tag{4.8}$$

which vanish along rod 1 and 3 respectively. It is obvious that both $\frac{\partial}{\partial\tilde{\phi}}$ and $\frac{\partial}{\partial\tilde{\psi}}$ generate orbits with period 2π. Their corresponding coordinates $(\tilde{\phi},\tilde{\psi})$ are related to (ϕ,ψ) by

$$\tilde{\phi} = \phi - \frac{1}{n}\,\psi\,, \qquad \tilde{\psi} = \frac{1}{n}\,\psi\,. \tag{4.9}$$

If we recall that m and n are given by (4.3), the metric (4.7) can be written in terms of these coordinates in either of the following two forms:

$$\begin{aligned}
\mathrm{d}s_{\mathrm{H}}^2 &= \frac{2\varkappa^2c^2}{(1-a^2)(1+cx)}\left[(1-c)\frac{\mathrm{d}x^2}{G(x)} + \frac{4a^2(1+x)}{g_1(x)}\,\mathrm{d}\tilde{\phi}^2 + (1-x)m^2g_1(x)(\mathrm{d}\tilde{\psi} + f_1(x)\,\mathrm{d}\tilde{\phi})^2\right],\\
\mathrm{d}s_{\mathrm{H}}^2 &= \frac{2\varkappa^2c^2}{(1-a^2)(1+cx)}\left[(1-c)\frac{\mathrm{d}x^2}{G(x)} + \frac{4a^2m^2(1-x)}{g_2(x)}\,\mathrm{d}\tilde{\psi}^2 + (1+x)g_2(x)(\mathrm{d}\tilde{\phi} + f_2(x)\,\mathrm{d}\tilde{\psi})^2\right].
\end{aligned} \tag{4.10}$$

Here, we have defined

$$\begin{aligned}
g_1(x) &= \frac{1-c}{1+c}(1+x) + a^2(1-x)\,, \qquad f_1(x) = \frac{(1-a^2)(1+x)}{m^2g_1(x)}\,,\\
g_2(x) &= (1-x) + a^2\frac{1+c}{1-c}(1+x)\,, \qquad f_2(x) = \frac{(1-a^2)(1-x)}{g_2(x)}\,.
\end{aligned} \tag{4.11}$$

Both metrics in (4.10) resemble that of a 3-sphere, albeit a squashed one, with degenerations occuring at $x=\pm1$. We can examine the vicinity of the "north pole" $x=-1$ by introducing the new coordinate r_1, such that $x=-1+r_1^2$. For small r_1, the first metric in (4.10) reduces to

$$\mathrm{d}s_{\mathrm{H}}^2 \to \frac{4\varkappa^2c^2}{(1-a^2)(1-c)}(\mathrm{d}r_1^2 + r_1^2\,\mathrm{d}\tilde{\phi}^2) + 2\varkappa^2(1-a^2)(1+c)n^2\,\mathrm{d}\tilde{\psi}^2\,. \tag{4.12}$$

On the other hand, we can examine the vicinity of the "south pole" $x=1$ by introducing the new coordinate r_2, such that $x=1-r_2^2$. For small r_2, the second metric in (4.10) reduces to

$$\mathrm{d}s_{\mathrm{H}}^2 \to \frac{4\varkappa^2 c^2(1-c)}{(1-a^2)(1+c)^2}(\mathrm{d}r_2^2 + m^2 r_2^2\,\mathrm{d}\tilde{\psi}^2) + 2\varkappa^2(1-a^2)(1+c)\frac{n^2}{m^2}\,\mathrm{d}\tilde{\phi}^2\,. \tag{4.13}$$

Note that there will be a conical singularity at the point $r_2 = 0$ unless $m = \pm 1$, which is consistent with our interpretation above. Thus, the local behaviour of the horizon metric at these two poles (modulo the possible presence of the conical singularity) is similar to the standard metric on S^3. However, there are some global differences resulting from (4.9). Since ψ has period 2π, it follows from (4.9) that identifications should be made under the operation:

$$(\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi} + 2\pi, \tilde{\phi})\,, \qquad (\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi} + \frac{2\pi}{n}, \tilde{\phi} - \frac{2\pi}{n})\,. \tag{4.14}$$

These are precisely the identifications of S^3 required to turn it into the lens space $L(n, 1)$ [38, 82], thus confirming that the horizon has this topology.

It is instructive to similarly examine the geometry of the space near the point $(x, y) = (1, -1)$, where rod 3 and rod 4 meet up. We now introduce two Killing vector fields $\frac{\partial}{\partial\phi'}$ and $\frac{\partial}{\partial\psi'}$, defined by

$$\frac{\partial}{\partial\phi'} = \ell_1 + n\ell_4\,, \qquad \frac{\partial}{\partial\psi'} = \ell_4\,, \tag{4.15}$$

which vanish along rod 3 and 4 respectively. It is obvious that both $\frac{\partial}{\partial\phi'}$ and $\frac{\partial}{\partial\psi'}$ generate orbits with period 2π. Their corresponding coordinates (ψ', ϕ') are related to (ϕ, ψ) by

$$\phi' = \phi\,, \qquad \psi' = \psi - n\phi\,. \tag{4.16}$$

If we introduce the new coordinates r and θ by

$$x = 1 - (1+c)\,r^2 \sin^2\theta\,, \qquad y = -1 - (1-c)\,r^2\cos^2\theta\,, \tag{4.17}$$

then the spatial part of (4.1) in the region of small r is

$$\mathrm{d}s^2 = \frac{\varkappa^2(1+c)[1-c-a^2(1+c)]}{1-a^2}\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + m^2\sin^2\theta\,\mathrm{d}\phi'^2 + \cos^2\theta\,\mathrm{d}\psi'^2\right)\right]. \tag{4.18}$$

This is just a flat-space geometry with a conical singularity along the $\theta = 0$ ($x = 1$) axis in general. The form of this geometry is exactly the same as that of the more familiar case of the black ring. In particular, there is no orbifold singularity at the origin $r = 0$. For a taking values in Range II however, this metric has the wrong signature, and is an indication of the fact that there are closed time-like curves (CTCs) in this region.

Now, the requirement for the absence of CTCs is that the 2×2 metric g_{ij}, $i, j = \psi, \phi$, be positive semi-definite. This is equivalent to checking that its determinant and one of the diagonal components, say $g_{\psi\psi}$, are non-negative. These two quantities can be read off from (4.1), and we have checked that the following results hold: For a taking values in Range I, there are no CTCs anywhere in the space-time outside the horizon. For a taking values in Range II, there are no CTCs outside the horizon and naked singularity; however, CTCs do exist in the region inside the naked singularity, as we have seen above in the vicinity of $r = 0$. Fortunately, they are not a concern for us as this region is not accessible to observers outside the naked singularity.

Finally, we note that the ADM mass, entropy (from area) and temperature (from surface gravity) of the black-lens horizon are given by the expressions:

$$M = \frac{3\pi\varkappa^2 c}{2G}\,, \quad S = \frac{4\pi^2\varkappa^3 c^2}{G}\sqrt{\frac{2}{(1-a^2)(1+c)}}\,, \quad T = \frac{1}{8\pi\varkappa c}\sqrt{2(1-a^2)(1+c)}\,. \tag{4.19}$$

It follows that the Smarr relation:

$$\frac{2}{3}M = TS\,, \tag{4.20}$$

is satisfied by the static black lens. When $\varkappa$ and c are kept fixed (so that mass M is fixed), observe from Fig. 4.2 that solutions in Range I have a value of a that increases with n. From (4.19), it follows that S also increases with n, so black lenses with larger n are entropically favoured. The configuration in this range with the highest entropy is the $n \to \infty$ limiting case of the Schwarzschild black hole. On the other hand, it can be seen that solutions in Range II have an entropy that decreases as n is increased. In this case, black lenses with smaller n are entropically favoured. Note also that solutions in Range II have higher entropy than solutions in Range I.

To summarize, the black lens solution (4.1) can be divided into two ranges I and II depending on the value of a, which exhibit rather different properties outside the horizon. All solutions in Range I possess a conical singularity along the $x = 1$ axis, but are otherwise regular and well-behaved. Included in this range are all positive values of n. For the case $n = 1$ [corresponding to $a = (1-c)/(1+c)$], we recover a black hole with $L(1,1) \cong S^3$ horizon topology, as we mentioned in the introduction, but with a conical singularity. This solution will be revisited in subsection 4.4.2. For the case $n = 2$, we have a black lens with $L(2,1) \cong \mathbb{R}P^3$ event-horizon topology.

Solutions in Range II also in general possess a conical singularity along the $x = 1$ axis, although it can be eliminated for a particular value of a in this range and with $n \geq 3$. However, all solutions in this range possess a naked singularity with spherical topology surrounding the point $(x, y) = (1, -1)$. A similar situation was found in the static black lens solution of [96], which actually contains two such singularities. It was conjectured in [96] that adding angular momentum might eliminate such singularities, and we shall revisit this issue in the following section when we add a single asymptotic angular momentum to our black lens solution.

4.3 Single-rotating black lens

We have successfully derived a single-rotating black lens solution using the inverse scattering method. The reader is referred to appendix A for more details of the construction. In C-metric type coordinates, the metric we obtain is

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{H(y,x)}{H(x,y)}(\mathrm{d}t - \omega_\psi\,\mathrm{d}\psi - \omega_\phi\,\mathrm{d}\phi)^2 - \frac{F(x,y)}{H(y,x)}\,\mathrm{d}\psi^2 + 2\frac{J(x,y)}{H(y,x)}\,\mathrm{d}\psi\mathrm{d}\phi \\ &\quad + \frac{F(y,x)}{H(y,x)}\,\mathrm{d}\phi^2 + \frac{\varkappa^2 H(x,y)}{2(1-a^2)(1-b)^3(x-y)^2}\left(\frac{\mathrm{d}x^2}{G(x)} - \frac{\mathrm{d}y^2}{G(y)}\right), \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} \omega_\psi &= \frac{2\varkappa}{H(y,x)}\sqrt{\frac{2b(1+b)(b-c)}{(1-a^2)(1-b)}}\,(1-c)(1+y)\big\{2[1-b-a^2(1+bx)]^2(1-c) \\ &\qquad - a^2(1-a^2)b(1-b)(1-x)(1+cx)(1+y)\big\}\,, \\ \omega_\phi &= \frac{2\varkappa}{H(y,x)}\sqrt{\frac{2b(1+b)(b-c)}{(1-a^2)(1-b)}}\,a(1-c)(1+x)^2(1+y)\big[a^4(1+b)(b-c) \\ &\qquad + a^2(1-b)(-b+cb+2c) - (1-b)^2 c\big]\,. \end{aligned} \tag{4.22}$$

The functions G, H, F and J are defined as

$$\begin{aligned} G(x) &= (1-x^2)(1+cx)\,, \\ H(x,y) &= 4(1-b)(1-c)(1+bx)\big\{(1-b)(1-c) - a^2[(1+bx)(1+cy) + (b-c) \\ &\qquad \times(1+y)]\big\} + a^2(b-c)(1+x)(1+y)\big\{(1+b)(1+y)[(1-a^2)(1-b)c \end{aligned}$$

$$
\begin{aligned}
& \times(1+x) + 2a^2 b(1-c)] - 2b(1-b)(1-c)(1-x)\big\} \,, \\
F(x,y) &= \frac{2\varkappa^2}{(1-a^2)(x-y)^2}\Big[4(1-c)^2(1+bx)[1-b-a^2(1+bx)]^2 G(y) - a^2 G(x) \\
& \times (1+y)^2\Big([1-b-a^2(1+b)]^2(1-c)^2(1+by) - (1-a^2)(1-b^2)\times \\
& \times (1+cy)\big\{(1-a^2)(b-c)(1+y) + [1-3b-a^2(1+b)](1-c)\big\}\Big)\Big] \,, \\
J(x,y) &= \frac{4\varkappa^2 a(1-c)(1+x)(1+y)}{(1-a^2)(x-y)}[1-b-a^2(1+b)][(1-b)c + a^2(b-c)]\times \\
& \times [(1+bx)(1+cy) + (1+cx)(1+by) + (b-c)(1-xy)] \,.
\end{aligned}
\tag{4.23}
$$

The coordinates take the same ranges as in the previous section, while the parameters satisfy $0 < c \leq b < 1$ and $-1 < a < 1$. Note that the metric is invariant under the action $a \to -a$ and $\phi \to -\phi$. The space-time is asymptotically flat if ψ and ϕ are identified with period 2π independently. The ADM mass and angular momenta of this space-time can be calculated to be

$$
M = \frac{3\pi\varkappa^2 b(1-c)}{2G(1-b)} \,, \qquad J_\psi = \frac{\pi\varkappa^3\sqrt{2(1-a^2)b(1+b)(b-c)}(1-c)}{G(1-b)^{3/2}} \,, \qquad J_\phi = 0 \,.
\tag{4.24}
$$

As desired, we have a space-time with angular momentum in a single direction, namely the ψ direction. The static limit is recovered when $b = c$, and (4.21) reduces to the previous solution (4.1).

The rod structure of the single-rotating black lens is qualitatively the same as that of the static black lens (4.1). The only differences are the directions of the second and third rod are respectively $\ell_2 = \frac{1}{\kappa}(1, \Omega_\psi, \Omega_\phi)$, with

$$
\begin{aligned}
\kappa &= \frac{1}{4\varkappa}\sqrt{\frac{2c(1-a^2)}{b(1+b)}}\frac{(1-b)^2(1+c)}{(1-c)[(1-b)c + a^2(b-c)]} \,, \\
\Omega_\psi &= \frac{1}{\varkappa}\sqrt{\frac{(1-b)(b-c)}{2(1-a^2)b(1+b)}}\frac{1}{1-c} \,,
\end{aligned}
$$

$$\Omega_\phi = \frac{1}{2\varkappa}\sqrt{\frac{(1-b)(b-c)}{2(1-a^2)b(1+b)}}\frac{a[1-b-a^2(1+b)]}{(1-b)c+a^2(b-c)}\,, \tag{4.25}$$

and $\ell_3 = (0, \frac{n}{m}, \frac{1}{m})$, with

$$n = \frac{2a[(1-b)c+a^2(b-c)]}{[1-b-a^2(1+b)](1-c)}, \quad m = \frac{(1-a^2)(1-b)(1+c)}{[1-b-a^2(1+b)](1-c)}\sqrt{\frac{(1-b)}{(1+b)}}. \tag{4.26}$$

Rod 1 and 4 as usual represent the two asymptotic axis, and their directions $\ell_1 = (0,0,1)$ and $\ell_4 = (0,1,0)$ are identified as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group to ensure asymptotic flatness.

Several special cases can immediately be read off from this result. One is when $a = 0$, in which rod 1 is parallel to rod 3. In this event, we recover the Emparan–Reall black ring with parameters b and c as introduced in [36]. The other is when $a = \pm\sqrt{(1-b)/(1+b)}$; in which rod 3 is joined up to rod 4 in the same direction. In this event, we recover the single-rotating five-dimensional Myers–Perry black hole (see subsection 4.4.2). The background limit is recovered when $b, c \to 0$ (see subsection 4.4.1).

To obtain a black lens with horizon topology $L(n, 1)$, we require that n is an integer. For simplicity we can assume n is positive; similar analysis applies if it is negative. It is again instructive to plot the n against a (for fixed b and c). The graph obtained is qualitatively similar to that in Fig. 4.2, except that now the vertical asymptotes are located at $a = \pm\sqrt{(1-b)/(1+b)}$. There are again two ranges of solutions to consider: the first $0 < a < \sqrt{(1-b)/(1+b)}$, which we call Range I; and the second $-1 < a < -\sqrt{(1-b)/(1+b)}$, which we call Range II. Note that in Range I, n takes integer values in the interval $(0, \infty)$; while in Range II, n takes integer values in the interval $(1, \infty)$.

Rod 2 at $y = -\frac{1}{c}$ represents the event horizon of the black lens. Ω_ψ and Ω_ϕ are the angular velocities of the event horizon along the two asymptotic axes parameterized by ψ and ϕ respectively. The ratio of these two quantities is, in fact, n/a^2. Thus, the event horizon of the black lens is rotating in both the ψ and ϕ directions, although not independently. This is in contrast to the situation at asymptotic infinity, in which only the angular momentum in the ψ-direction survives.

Let us now turn to a study of possible conical singularities in the space-time. The analysis is very similar to the case of static black lens: we require $m = \pm 1$; otherwise, the space-time contains a conical singularity along rod 3. As in the static case, it is possible to show that the condition $m = \pm 1$ cannot be satisfied for any a in Range I. Indeed, solving the equation for n in (4.26) in terms of b or c and substituting it into m shows that $m^2 > (n+a)^2$.

Thus, the only possibility for the conical singularity to be eliminated lies in Range II. It turns out to be simpler to solve the two conditions (1) n is a positive integer and (2) $m = \pm 1$ simultaneously in terms of (b, c), rather than (a, b) or (a, c). We obtain the solution:

$$b = \frac{n(n+2a)}{n^2 + 2na + 2a^2}\,, \qquad c = \frac{n(1 - n^2 - 3na - 3a^2)}{(n+2a)(1 - n^2 - na - a^2)}\,. \tag{4.27}$$

The requirement that $b \geq c$ then implies that a takes values in the range

$$\frac{\sqrt{n^2 - 4} - n}{2} \leq a < 0\,, \tag{4.28}$$

with the lower bound corresponding to the static case $b = c$. Note that unlike the static case, the $n = 2$ solution is a valid one when there is rotation. The static limit of this particular solution forces $b, c \to 0$, and the black lens disappears leaving just the background space-time (see subsection 4.4.1).

Unfortunately, all solutions in Range II suffer from the same pathology as in the static case; namely, the value of $H(x,y)$ vanishes on a closed surface of spherical topology that surrounds the point $(x,y)=(1,-1)$, separating it from the rest of the space-time, including the horizon and asymptotic infinity. On this surface, the curvature invariant $R_{abcd}R^{abcd}$ diverges, so it is a nakedly singular one. This singularity does not exist in Range I. For either range, if we extend the coordinate range below the horizon $y<-1/c$, there is also a curvature singularity at $y\to-\infty$, $x\to-1$.

When rotation is present, there will be an ergoregion in the space-time where the Killing vector $\partial/\partial t$ changes from being time-like to space-like. It is bounded by the closed surface on which the value of $H(y,x)$ vanishes. For solutions in Range I, it is possible to show that this surface completely encloses the event horizon, and only intersects rod 1 and rod 3. Thus, this surface has the same $L(n,1)$ topology as the event horizon. For solutions in Range II with sufficiently small values of a^2, there will continue to be an ergoregion with surface topology $L(n,1)$ enclosing the event horizon. However, another separate ergoregion appears enclosing the naked singularity, with an S^3 surface topology since it intersects rod 3 and rod 4. For larger values of a^2 (which includes the case when the conical singularity is eliminated), the two ergoregions in fact merge into a single one that encloses both the event horizon and the naked singularity, as well as the finite axis rod 3. Its surface intersects rod 1 and rod 4, and so it has an S^3 topology.

Now, it is possible to analyse the horizon geometry as was done for the static case, to verify that it has an $L(n,1)$ lens-space topology. The details are similar; in particular, the required transformation has the same form as (4.9). We will not repeat the analysis here. Similarly, it can be shown that the geometry near the point $(x, y) = (1, -1)$ is just flat space with a conical singularity along the $x = 1$ axis. However, when a takes values in Range II, there appears to be CTCs in this region.

We can check the possible existence of CTCs in the space-time (4.21), in the same manner as the static case. Since the expressions are a lot more complicated in this case, we have resorted to numerical analysis to do so. When a takes values in Range I, no CTCs were found outside the horizon despite extensive checks. However, when a takes values in Range II, CTCs were found not only inside the naked singularity, but also outside it when there is rotation present. These CTCs tend to occur very close to the surface of the naked singularity, and so appear to be more a feature of the naked singularity rather than the black lens itself.

Finally, we note the following expressions for the entropy and temperature of the black-lens horizon:

$$\begin{aligned} S &= \frac{4\pi^2\varkappa^3}{G}\sqrt{\frac{2bc(1+b)}{1-a^2}}\,\frac{(1-c)[(1-b)c+a^2(b-c)]}{(1-b)^2(1+c)}\,, \\ T &= \frac{1}{8\pi\varkappa}\sqrt{\frac{2c(1-a^2)}{b(1+b)}}\,\frac{(1-b)^2(1+c)}{(1-c)[(1-b)c+a^2(b-c)]}\,. \end{aligned} \tag{4.29}$$

It can then be checked that the rotating black lens satisfies the Smarr relation:

$$\frac{2}{3}M = TS + \Omega_\psi J_\psi + \Omega_\phi J_\phi\,. \tag{4.30}$$

We also note that the Komar mass and angular momenta evaluated at the horizon of the black lens agree with the asymptotic quantities in (4.24). This implies that the conical singularity and/or naked singularity do not contribute to the total ADM mass and angular momentum of the space-time.

To summarize, the black lens solution (4.21) can be divided into two ranges I and II, as defined above, which exhibit different properties. All solutions in Range I possess a conical singularity along the $x = 1$ axis, but are otherwise regular and well-behaved. Included in this range are all positive values of n. For the case $n = 1$, we recover a rotating black hole with $L(1, 1) \cong S^3$ horizon topology, with a conical singularity attached to it. This solution will be revisited in subsection 4.4.2. For the case $n = 2$, we have a rotating black lens with $L(2, 1) \cong \mathbb{R}P^3$ horizon topology.

Solutions in Range II also in general possess a conical singularity along the $x = 1$ axis, although it can be eliminated for a particular value of a in this range and with $n \geq 2$. However, all solutions in this range possess a naked singularity with spherical topology surrounding the point $(x, y) = (1, -1)$. Thus, the introduction of a single rotation to the black lens does not remove this singularity, as was hoped for in [96]. Moreover, the rotation causes CTCs to appear just outside the surface of the naked singularity. If one does not desire the presence of CTCs with its associated paradoxes, then it would appear that Range I solutions are the more appropriate ones to consider.

4.4 Background space-time and black-hole limit

In this section, we study some special limits of the black lens solutions, including the background space-times and the black-hole limits. Again, without loss of generality, we consider the case when $n > 0$; similar analysis applies when $n < 0$.

4.4.1 Background space-time

The background space-time for the static black lens (4.1), or more generally the rotating black lens (4.21), can be uniquely determined after making the following reasonable assumptions: Firstly, we should take $c \to 0$ to eliminate the horizon; in the context of Fig. 4.1, this corresponds to taking the limit $z_1 \to z_2$ such that the second rod vanishes. Secondly, the mass and angular momentum of the background space-time should vanish. From the relevant expressions in (4.24), it follows that we should also take $b \to 0$ while keeping $b \geq c$. On the other hand, the non-negative integer n should be fixed to maintain the event horizon topology. For solutions in Range I, it follows from the equation for n in (4.26) that a takes the form

$$a = 1 - \frac{1+n}{n}\, b + O(b^2)\,, \tag{4.31}$$

in this limit. The black lens metric (4.21) then reduces to

$$\begin{aligned} \mathrm{d}s^2 = -\mathrm{d}t^2 \quad + \quad & \frac{\varkappa^2}{(1+n)(x-y)^2}\Bigg\{[2-n(x+y)]\left(\frac{\mathrm{d}x^2}{1-x^2}-\frac{\mathrm{d}y^2}{1-y^2}\right) \\ & +\frac{1-x^2}{2-n(x+y)}\left[(2+n(1-y))\mathrm{d}\phi - n(1+y)\mathrm{d}\psi\right]^2 \\ & -\frac{1-y^2}{2-n(x+y)}\left[(2+n(1-x))\mathrm{d}\psi - n(1+x)\mathrm{d}\phi\right]^2\Bigg\}\,, \end{aligned} \tag{4.32}$$

with $-\infty < t < \infty$, $-1 \leq x \leq 1$ and $-\infty < y \leq -1$. ψ and ϕ are identified with period 2π independently to ensure asymptotic flatness. Note that this background depends only on the parameters $\varkappa$ and n, as expected. It has a non-vanishing curvature if $n \neq 0$.

It can be verified that this space-time contains three axes: the two usual semi-infinite axes represented by rod 1 at $x = -1$ and rod 3 at $y = -1$, and a finite one represented by rod 2 at $x = 1$ with direction $(0, \frac{n}{1+n}, \frac{1}{1+n})$. It is clear that in general there is a conical singularity along the finite axis. On the other hand, we have checked that there are no CTCs present in this space-time.

The two turning points in the space-time deserve more attention. The region around the second turning point was already examined in section 4.2, and the details are largely similar in this case. The first turning point is where the $x = -1$ and $x = 1$ axes meet up ($y \to \infty$). Let us now examine the region around this point. We change to new coordinates (r, θ) as follows:

$$x = \cos 2\theta\,, \qquad y = -\frac{2}{r^2}\,, \tag{4.33}$$

where $0 \leq \theta \leq \pi/2$. The point in question is then located at $r = 0$. For small r, the spatial part of the metric (4.32) becomes

$$\mathrm{d}s^2 = \frac{2\varkappa^2 n}{1+n}\left\{\mathrm{d}r^2 + r^2\left[\mathrm{d}\theta^2 + \frac{(1+n)^2}{n^2}\sin^2\theta\,\mathrm{d}\psi^2 + \cos^2\theta\left(\mathrm{d}\phi - \frac{1}{n}\,\mathrm{d}\psi\right)^2\right]\right\}, \tag{4.34}$$

which is just a flat-space geometry. To see this, we may introduce azimuthal coordinates $\tilde{\phi}$ and $\tilde{\psi}$, defined by

$$\tilde{\phi} = \phi - \frac{1}{n}\,\psi\,, \qquad \tilde{\psi} = \frac{1}{n}\,\psi\,, \tag{4.35}$$

such that the Killing vector fields $\frac{\partial}{\partial\tilde{\psi}}$ and $\frac{\partial}{\partial\tilde{\phi}}$ vanish at $\theta = 0$ and $\theta = \pi/2$ respectively. When $n \geq 2$, this transformation does not have unit determinant, and it follows that there is a $\mathbb{Z}_n$ orbifold singularity at $r = 0$. There is no orbifold singularity when $n = 1$, which can also be seen from the fact that in this case, the two points where the axes meet up are mirror images of each other. In general, there is also a conical singularity [with excess angle $2\pi n/(1+n)$] along the $\theta = 0$ axis.

On the other hand, for solutions in Range II, it follows from the equation for n in (4.26) that a should take the form

$$a = -1 + \frac{n-1}{n}\, b + O(b^2)\,, \tag{4.36}$$

in the background limit. In this case, the black lens metric (4.21) then reduces to

$$\begin{aligned} \mathrm{d}s^2 = -\mathrm{d}t^2 \quad + \quad & \frac{\varkappa^2}{(1-n)(x-y)^2}\left\{ [2+n(x+y)] \left(\frac{\mathrm{d}x^2}{1-x^2} - \frac{\mathrm{d}y^2}{1-y^2} \right) \right. \\ & + \frac{1-x^2}{2+n(x+y)} \left[(2-n(1-y))\mathrm{d}\phi - n(1+y)\mathrm{d}\psi \right]^2 \\ & \left. - \frac{1-y^2}{2+n(x+y)} \left[(2-n(1-x))\mathrm{d}\psi - n(1+x)\mathrm{d}\phi \right]^2 \right\} . \end{aligned} \tag{4.37}$$

This background is actually related to the previous one (4.32) under the transformation $n \to -n$ and either $\psi \to -\psi$ or $\phi \to -\phi$. It has similar rod structure as (4.32) except now we have $\ell_2 = (0, \frac{n}{n-1}, \frac{1}{n-1})$. Again there is a $\mathbb{Z}_n$ orbifold singularity at the first turning point. There is in general a conical singularity along rod 2 at $x = 1$, as well as a naked singularity with spherical topology located at points where $2 + n(x + y) = 0$. It can be checked that there are no CTCs outside this naked singularity.

Now, if we restrict ourselves to the particular solution (4.27) in Range II which does not contain conical singularities, then we would require $m = \pm 1$ to hold even in the background limit. This can be seen to happen only when $n = 2$.[3] As mentioned in section 4.3, this corresponds to taking the static limit of this solution. However, there is still the spherical naked singularity in this background.

[3]Another way to see this is to plot c, as given in (4.27), against a in the range (4.28). For $n > 2$, the graphs do not touch the $c = 0$ axis at any point.

4.4.2 Black-hole limit

There are three limits in which black holes with spherical event horizon topology can be obtained from our black lens solution. In the interest of generality, we will only consider limits of the rotating black lens solution (4.21) here. The black-hole limits of the static black lens can readily be obtained as special cases.

First, consider the case when $a = \pm\sqrt{(1-b)/(1+b)}$, in which rod 3 at $x = 1$ is joined up to rod 4 at $y = -1$ in the same direction. To show that the metric (4.21) is equivalent to the five-dimensional Myers–Perry black hole, we need to first transform to Weyl–Papapetrou coordinates, and then to prolate spheroidal coordinates. The relevant formulae can be found in the appendices of [36]. The coordinate transformation relating the C-metric coordinates (x, y) to the prolate spheroidal coordinates $(\tilde{x}, \tilde{y})$ is

$$\begin{aligned} x &= \frac{(1-c)R_1 - (1+c)R_2 - 2R_3 + 2(1-c^2)\varkappa^2}{(1-c)R_1 + (1+c)R_2 + 2cR_3}\,, \\ y &= \frac{(1-c)R_1 - (1+c)R_2 - 2R_3 - 2(1-c^2)\varkappa^2}{(1-c)R_1 + (1+c)R_2 + 2cR_3}\,, \end{aligned} \tag{4.38}$$

where

$$R_1 = c\varkappa^2(\tilde{x}+\tilde{y})\,, \quad R_2 = c\varkappa^2(\tilde{x}-\tilde{y})\,, \quad R_3 = \varkappa^2\sqrt{c^2(\tilde{x}^2-1)(1-\tilde{y}^2)+(c\tilde{x}\tilde{y}-1)^2}\,. \tag{4.39}$$

Finally, we transform to Boyer–Lindquist coordinates (r, θ):

$$\tilde{x} = \frac{2r^2}{\mu - \alpha^2} - 1\,, \qquad \tilde{y} = \cos 2\theta\,, \tag{4.40}$$

where μ and α are new parameters, related to b and c by

$$b = \frac{\mu}{4\varkappa^2 + \alpha^2}\,, \qquad c = \frac{\mu - \alpha^2}{4\varkappa^2}\,. \tag{4.41}$$

Under these transformations, (4.21) becomes

$$\begin{aligned} ds^2 &= -\frac{r^2-\mu+\alpha^2\cos^2\theta}{r^2+\alpha^2\cos^2\theta}\left(dt+\frac{\mu\alpha\sin^2\theta}{r^2-\mu+\alpha^2\cos^2\theta}\,d\psi\right)^2+r^2\cos^2\theta\,d\phi^2 \\ &\quad +(r^2+\alpha^2\cos^2\theta)\left(\frac{dr^2}{r^2-\mu+\alpha^2}+d\theta^2+\frac{r^2-\mu+\alpha^2}{r^2-\mu+\alpha^2\cos^2\theta}\,\sin^2\theta\,d\psi^2\right), \end{aligned} \tag{4.42}$$

which is the familiar form of the five-dimensional Myers–Perry black hole [18] rotating in the ψ direction.

The second limit we shall consider is when the finite space-like rod in the black lens space-time is shrunk to zero length — $z_2 \to z_3$ in the context of Fig. 4.1 — while preserving its Harmark direction. It turns out that we recover the five-dimensional Myers–Perry black hole only when a takes values in Range I, which we recall contains the black ring as a limiting case. Indeed, the transformation we seek is similar to the one used in the black ring case [71]: We define the parameters μ and α by

$$\mu=\frac{4\varkappa^2}{1-c}\,,\qquad \alpha^2=4\varkappa^2\frac{b-c}{(1-c)^2}\,, \tag{4.43}$$

such that they remain finite in the limit $b,c\to 1$ and $\varkappa\to 0$. In this limit, the relevant root of the equation for n in (4.26) has the form

$$a=\frac{n}{2}(1-c)+O(1-c)^2. \tag{4.44}$$

If we transform to new coordinates r and θ via the relations

$$\begin{aligned} x &= -1+\left(1-\frac{\alpha^2}{\mu}\right)\frac{4\varkappa^2\cos^2\theta}{r^2-(\mu-\alpha^2)\cos^2\theta}\,, \\ y &= -1-\left(1-\frac{\alpha^2}{\mu}\right)\frac{4\varkappa^2\sin^2\theta}{r^2-(\mu-\alpha^2)\cos^2\theta}\,, \end{aligned} \tag{4.45}$$

and rescale t:

$$t\to\sqrt{\frac{4\varkappa^2}{\mu-\alpha^2}}\;t\,, \tag{4.46}$$

then it can be checked that the metric (4.21) indeed reduces to the five-dimensional Myers–Perry metric (4.42), up to an overall constant factor.

The third limit in which a black hole can be obtained from the black lens solution is when $n = 1$. As we mentioned in the introduction, in this case, the topology of the horizon is also an S^3. Unfortunately we find that there is now a conical singularity attached to the black hole along rod 3. One way to see this is to push rod 4 to infinity by making rod 3 infinitely long, while preserving the latter's Harmark direction. In the context of Fig. 4.1, this corresponds to taking $z_3 \to \infty$. We first perform the transformations (4.38) to (4.41), and then take the limits $b, c \to 0$ and $\varkappa \to \infty$ such that $b\varkappa^2$ and $c\varkappa^2$ remain finite. In this limit, the relevant root of the equation for n in (4.26) has the form

$$a = 1 - \frac{(1+n)\mu}{4n\varkappa^2} + O\left(\frac{1}{\varkappa^4}\right), \tag{4.47}$$

where we keep n general for the time being. If we rescale $t \to \sqrt{n/(1+n)}\, t$, then (4.21) becomes, up to an overall constant factor,

$$\begin{aligned} \mathrm{d}s^2 \;=\; & -\frac{r^2 - \mu + \alpha^2\cos^2\theta}{r^2 + \alpha^2\cos^2\theta}\left(\mathrm{d}t + \frac{\mu\alpha\sin^2\theta}{r^2 - \mu + \alpha^2\cos^2\theta}\frac{1+n}{n}\,\mathrm{d}\psi\right)^2 \\ & + r^2\cos^2\theta\left(\mathrm{d}\phi - \frac{1}{n}\,\mathrm{d}\psi\right)^2 + (r^2 + \alpha^2\cos^2\theta)\times \\ & \left(\frac{\mathrm{d}r^2}{r^2 - \mu + \alpha^2} + \mathrm{d}\theta^2 + \frac{r^2 - \mu + \alpha^2}{r^2 - \mu + \alpha^2\cos^2\theta}\frac{(1+n)^2}{n^2}\,\sin^2\theta\,\mathrm{d}\psi^2\right). \end{aligned} \tag{4.48}$$

Now, we may introduce azimuthal coordinates $\tilde{\phi}$ and $\tilde{\psi}$, defined by

$$\tilde{\phi} = \phi - \frac{1}{n}\,\psi\,, \qquad \tilde{\psi} = \frac{1}{n}\,\psi\,, \tag{4.49}$$

such that the Killing vector fields $\frac{\partial}{\partial\tilde{\psi}}$ and $\frac{\partial}{\partial\tilde{\phi}}$ vanish at $\theta = 0$ and $\theta = \pi/2$ respectively. In these coordinates, the space-time described by (4.48) for the case

$n = 1$ can be seen to be just the five-dimensional Myers–Perry black hole rotating in the $\tilde{\psi}$ direction, but with a conical singularity (with excess angle π) along the $\theta = 0$ axis. For $n \geq 2$ however, this space-time is quotiented by $\mathbb{Z}_n$, so that (4.48) describes a black lens with horizon topology $L(n, 1)$ rotating in the $\tilde{\psi}$ direction, asymptotic to a locally flat space-time whose spatial sections are also lens spaces $L(n, 1)$. Note that there is still a conical singularity [with excess angle $2\pi n/(1+n)$] along the $\theta = 0$ axis.

4.5 Discussion

The main results of this chapter are as follows: We have derived the metric for an asymptotically flat black lens with $L(n, 1)$ event-horizon topology, with asymptotic angular momentum in one direction. Unfortunately, we have found that this space-time cannot be made completely regular. One either has to have a conical singularity attached to the black-lens event horizon, or a spherical naked singularity away from the event horizon. The latter interpretation was adopted by Evslin [96], who argued that since this singularity is isolated from the event horizon, it may somehow be eliminated locally without affecting the black lens itself. One of the results we have found is that introducing a single angular momentum does not seem able to eliminate it.

An obvious extension of this study would be to include angular momentum in the other azimuthal direction. We have in fact used the inverse scattering method to construct a black lens solution with two independent angular momenta (the construction is sketched in appendix A). It contains, as special cases, both the double-rotating black ring [49] as well as the single-rotating black lens (4.21). Unfortunately, this solution has a very complicated form which we will not present here. However, we have analysed its properties numerically, and it does not seem to be possible to eliminate the conical and naked singularities simultaneously, while at the same time maintaining a positive ADM mass for the black lens.

It may well turn out that completely regular black lenses do not exist, and that either conical or naked singularities are unavoidable. Of these two possibilities, we actually prefer the scenario containing conical singularities. The presence of a conical singularity (which in this case can be seen to be a conical excess, corresponding to what is also known as a strut singularity) is physically needed to balance the gravitational self-attraction of the black lens, something that the centrifugal force from the rotation seems unable to do alone. Such conical singularities are also quite common in other black hole solutions in general relativity, especially in space-times containing multiple black holes, all of which are considered legitimate space-times.

There are other reasons to prefer the scenario with conical singularities rather than naked singularities. As we have found, the introduction of angular momentum to the space-time causes closed time-like curves to appear near the naked singularity, but otherwise seem to be absent in those solutions without naked singularities. The class of black lens solutions containing just conical singularities (Range I as defined above) also has the feature that the familiar black ring solution emerges as a limiting case, and so can be regarded as the natural generalization of the black ring. It follows that this class of solutions possesses some appealing properties in common with the black ring solution, such as the existence of a well-defined black hole limit (see subsection 4.4.2).

What about black lenses with more general event-horizon topology $L(p,q)$, where p,q are coprime integers? Actually, this solution is still given by (4.21), if we set $n=p/q$ in (4.26). The condition for there to be no conical singularities along $x=1$ is then $m=1/q$, instead of $m=1$ as in (4.26), so that the direction of the finite space-like rod is actually $\ell_3=(p,q)$. It turns out that the analysis in section 4.3 is still valid in this case, and that there will be either a conical singularity or a naked singularity [with topology $L(q,p)$] in the space-time. Furthermore, there will be a $\mathbb{Z}_q$ orbifold singularity at the point $(x,y)=(1,-1)$ when $q>1$. This orbifold singularity may be resolved by introducing a second black lens with horizon topology $L(q,p)$ at this point. In fact, we have been able to construct various double black lens configurations [75], all of which, as expected, have conical and/or naked singularities.

In a recent paper [82], Lü et al. actually rediscovered and provided an alternative interpretation of the metric (4.1) which has no conical or naked singularities. They were able to satisfy the regularity conditions we studied in subsection 3.1.2, by giving up the condition of asymptotic flatness: that ψ and ϕ have period 2π. So (ℓ_1,ℓ_4) are not required to be identified with the pair of independent 2π-periodic generators of the $U(1)\times U(1)$ isometry group. What they found was a static black lens solution with horizon topology $L(n,m)$, asymptotic to a locally flat space-time whose spatial sections are in fact lens spaces $L(m,n)$. We expect that the metric (4.21) can similarly be interpreted as a rotating $L(n,m)$ black lens with an $L(m,n)$ asymptotic structure, for appropriate reidentifications of ψ and ϕ.

It is also possible to consider charged versions of our rotating black lens solution, for example in the context of five-dimensional minimal supergravity. Such a solution can be constructed using standard solution-generating techniques (see, e.g., [97]), and would be expected to carry both an electric charge and a (non-conserved) magnetic dipole charge. It would be interesting to construct and examine the properties of this solution. However, we note, somewhat disappointedly, that asymptotically flat supersymmetric black lenses have been proved not to exist in five-dimensional minimal supergravity [48].

At this point, one may be tempted to ponder about asymptotically flat black holes with non-spherical horizon topologies in six or higher dimensions. Unfortunately, most of the methods relied upon in this chapter: the generalized Weyl formalism and the concept of rod structures, the inverse scattering method, etc., will no longer be applicable. This is due to the simple fact that black holes in six or higher dimensions do not have the requisite number of commuting Killing vectors, and complete integrability of the Einstein equations is no longer assured. Thus, finding higher-dimensional analogues of black rings or black lenses would probably require a radically different approach. This is certainly a worthwhile and challenging problem to be left for the future.

Chapter 5

Rod-structure classification of gravitational instantons with $U(1) \times U(1)$ isometry

The rod-structure formalism has played an important role in the study of black holes in $D = 4$ and 5 dimensions with $\mathbb{R} \times U(1)^{D-3}$ isometry. In this chapter, we apply this formalism to the study of four-dimensional gravitational instantons with $U(1) \times U(1)$ isometry, which could serve as spatial backgrounds for five-dimensional black holes. As we have already mentioned, when the black holes are removed, together with the time dimension, the rod structure and in particular the regularity conditions, defined and studied in section 3.1, are applicable to gravitational instantons. Requiring the absence of conical and orbifold singularities will in general impose periodicity conditions on the coordinates, and we illustrate this by considering known gravitational instantons in this class. Some previous results regarding certain gravitational instantons are clarified in the process. Finally, in this chapter, we show how the rod-structure formalism is able to provide a classification of gravitational instantons, and speculate on the existence of possible new gravitational instantons.

5.1 Introduction

In all the known five-dimensional asymptotically flat completely regular vacuum black hole space-times, when the black holes/rings are removed, the resulting background space-times are a direct product of four-dimensional flat space and a flat time dimension [18, 19, 50–54]. Another possible background space-time that has been considered more recently in the literature [93, 98, 99] is the direct product of Euclidean self-dual Taub-NUT space [100, 101] and a flat time dimension. It turns out that four-dimensional flat space and the self-dual Taub-NUT space are but the simplest examples of gravitational instantons, which are defined to be non-singular four-dimensional Euclidean solutions to the Einstein equations [102]. They were extensively studied in the late 1970's and early 1980's within the context of Euclidean quantum gravity (see, e.g., [103]).

In the present context, these gravitational instantons will be Ricci-flat and have a $\mathcal{T} = U(1)\times U(1)$ isometry group. It turns out that many of the known gravitational instantons fall into this class of manifolds.[1] Besides the above-mentioned two examples, we also have the Euclidean Schwarzschild and Kerr instantons [104], the Eguchi–Hanson instanton [105], the Taub-bolt [106] and Kerr-bolt [107] instantons, and the multi-Taub-NUT [101] and Gibbons–Hawking [108] instantons when all the so-called nuts are collinear. For such gravitational instantons, it is possible to define the rod structure and analyze the regularity conditions as has been done in section 3.1 for five-dimensional black holes. This is because we can add a flat time dimension to these gravitational instantons to obtain five-dimensional space-times with $\mathbb{R} \times U(1)^2$ isometry, whose rod structures are essentially of the corresponding gravitational instantons, since the flat time dimension plays a trivial role.

For these gravitational instantons, all rods in their rod structures are space-like and represent (two-dimensional) rotational axes, and the turning points are the (zero-dimensional) intersection points of these axes. The directions of any two adjacent space-like rods are identified as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group, and all of these pairs must be related by $GL(2, \mathbb{Z})$ transformations. The orbit space $\hat{I} = I/\mathcal{T}$ of these manifolds I is similar to that of the black hole space-times with isometry group $\mathcal{G} = \mathbb{R} \times \mathcal{T}$ analyzed in section 3.1, i.e., $\hat{I} = \{z + i\rho \in \mathbb{C} \mid \rho > 0\}$, except that there are no boundary segments representing the horizons, and no boundary corners representing points where horizons intersect with axes. With all the axes removed, the isometry group $\mathcal{T} = U(1) \times U(1)$ naturally gives these manifolds a $\mathcal{T}$-principal fibre bundle structure over the base space $\hat{I}$.

[1] It may well be that the $U(1) \times U(1)$ is only a subgroup of a larger isometry group that the gravitational instanton possesses. But for our purpose, the existence of a $U(1) \times U(1)$ isometry subgroup is a sufficient condition.

For each of the known gravitational instantons in the class considered here, we will calculate its rod structure and check the conditions for the space to be free of conical and orbifold singularities. In most cases, this is a straightforward calculation which gives the required identifications of the coordinates. For the case of the self-dual Taub-NUT instanton, we will obtain the same identifications that Misner [79] famously found by considering two overlapping coordinate systems to eliminate Dirac-type string singularities. However, our approach has the advantage that it can be applied to other gravitational instantons, no matter how complicated, in a systematic and conceptually unified way. For example, we will be able to derive the required identifications for the multi-collinearly-centered Taub-NUT instanton; something which has not always been done correctly in the literature. Unfortunately, we will also find that the Kerr-bolt solution [107, 109] is not a true gravitational instanton in the sense that it can never be made completely regular; although appropriate identifications of the coordinates will eliminate the conical singularities, there will always be orbifold singularities at the two fixed points of the $U(1) \times U(1)$ isometry.[2]

Our consideration of the various examples will also illustrate the potential as well as limitations of using the rod structure as a way to distinguish different gravitational instantons. On one hand, the rod structure used here is able to distinguish the Eguchi–Hanson and Taub-bolt instantons, whereas the traditional (Harmark) rod structure [36, 37] is unable to do so. On the other hand, we will see that our rod

[2]The presence of orbifold singularities is sometimes tolerated in modern contexts such as string theory. However, we shall adopt the more traditional viewpoint that regular manifolds should strictly be free of such singularities.

structure is unable to say, tell the presence of NUT charge. For example, the self-dual Taub-NUT instanton has the same rod structure as that of four-dimensional flat space (in appropriate coordinates), while the double-centered Taub-NUT instanton has the same rod structure as the Eguchi–Hanson instanton. This is a manifestation of the fact that, in the terminology to be introduced in section 5.2, the rod structure is unable to distinguish between asymptotically locally flat gravitational instantons, and their asymptotically Euclidean or asymptotically locally Euclidean counterparts.

Despite these limitations, the rod-structure formalism turns out to be a useful way to classify gravitational instantons with $U(1) \times U(1)$ isometry. We will show how imposing the requirement that adjacent pairs of rod directions are related by $GL(2, \mathbb{Z})$ transformations will lead to restrictions on the directions that the various rods can take. For a given number of turning points, this will allow us to list down all possible rod structures that would correspond to regular manifolds without conical or orbifold singularities if appropriate identifications are made. This is done explicitly for the case of two and three turning points. Although some of them can be associated to known gravitational instantons, there is a countable infinity of new rod structures which can not. It is likely that other considerations (e.g., topological constraints or existence of curvature singularities) will rule out many of these new rod structures. Nevertheless, it is tantalizing to wonder if at least some of them would be associated to as yet undiscovered gravitational instantons.

This chapter is organized as follows: A brief review of some relevant aspects of gravitational instantons is given in section 5.2. In section 5.3, the rod structures of various known gravitational instantons with $U(1)\times U(1)$ isometry are analyzed, and the regularity conditions are checked for each case. Some previous results regarding certain of these gravitational instantons are clarified in the process. In section 5.4, we show how the regularity conditions can in principle be used to determine all allowed rod structures that could be associated to gravitational instantons, and speculate on the existence of possible new gravitational instantons. The chapter ends with a discussion of some open questions and possible extensions of this work.

5.2 Review of gravitational instantons

Gravitational instantons are defined as non-singular four-dimensional Euclidean solutions to the Einstein equations [102]. As gravitational analogues of Yang–Mills instantons, they are stationary phase points of the path integral in Euclidean quantum gravity, and provide tunnelling amplitudes between topologically distinct gravitational vacua. When the cosmological constant $\Lambda = 0$, gravitational instantons are nothing but Ricci-flat Riemannian 4-manifolds. Reviews on gravitational instantons may be found in [110–112].

Gravitational instanton symmetries have been classified by Gibbons and Hawking [102]. In their paper, one-parameter isometry group actions of a gravitational instanton were classified by their two possible types of fixed points: isolated points called nuts and 2-surfaces called bolts. For example, the isometry parameterized by the Euclidean time coordinate has a bolt at the (Euclidean) horizon for the

Euclidean Schwarzschild instanton; while it has two isolated nuts, respectively located at the two poles of the (Euclidean) horizon, for the Euclidean Kerr instanton with nonzero rotation parameter. But often the isometry group of a gravitational instanton is more than one-dimensional, and one has to pick out a particular one-parameter isometry subgroup, to see whether its fixed points are nuts or bolts. If the gravitational instanton admits a $U(1) \times U(1)$ isometry group, there are in fact infinitely many possible choices of this one-parameter isometry subgroup, and the corresponding fixed-point set depends on this choice.

Recall that in the rod structure of a gravitational instanton with $U(1) \times U(1)$ isometry, we have the property that along a (space-like) rod, its associated Killing vector field vanishes. So a rod represents a two-dimensional fixed-point set, and thus a bolt, for its associated Killing vector field. At a turning point where two adjacent rods intersect, both the Killing vector fields associated to the two rods vanish. Thus a turning point is a fixed point for the whole $U(1) \times U(1)$ isometry group. It represents a nut for any Killing vector field that generates isometries in the $U(1) \times U(1)$ group, provided it is linearly independent with each of the directions of the two adjacent rods. Moreover, there are no nuts or bolts away from the rods and turning points, with $\rho > 0$. Hence, it will be clear where the nut and bolt fixed points will be for any Killing vector field which generates isometries in the $U(1) \times U(1)$ group, once we know the rod structure of the gravitational instanton.

As mentioned in subsection 3.1.2, the gravitational instantons, as manifolds with $U(1) \times U(1)$ isometry, are uniquely determined by the rod structure, so it is natural to relate their topological invariants to their rod structures. It turns out that the

Euler number of these gravitational instantons can be easily read off from the rod structure. The Euler number of a compact manifold M with a one-parameter isometry group is $\chi[M] = \sum_{\text{bolts}} \chi_i + \sum_{\text{nuts}} 1$, where the bolts and nuts are all referred with respect to that one-parameter isometry group, and χ_i is the Euler number for the i-th bolt [102, 107]. The result does not depend on choice of the one-parameter isometry group. This formula also holds for manifolds with boundary provided that the Killing vector field corresponding to the one-parameter isometry group is either everywhere tangential or is everywhere transverse to the boundary. The gravitational instantons considered in this chapter satisfy this condition. For a compact bolt in the current context, it is easy to show that it will always have topology S^2, and so have Euler number 2. Then we can see that the Euler number for a gravitational instanton with $U(1) \times U(1)$ isometry is nothing but the number of turning points in the rod structure.

We note that it may also be possible to read off the Hirzebruch signature $\tau[M]$ from the rod structure, using the results of [102, 107]. These results rely upon defining the type (p, q) of a nut for a one-parameter isometry group as was done in [102]. Now, it is possible to calculate the (p, q)-type of each nut from the rod structure.[3] However, there are certain subtleties involving the boundary terms in the formula for $\tau[M]$, and we will not discuss the computation of the Hirzebruch signature nor the (p, q)-nut-type formalism any further here.

[3]This can be done by constructing consistently oriented coordinate charts near the rods and turning points, and requiring appropriate relations between these local coordinate charts and the directions of rods. By doing this, it is actually possible to fix the directions of all the rods up to an overall sign.

All the explicitly known gravitational instantons with $\Lambda = 0$, except for T^4 with a flat metric, are non-compact with "infinities". It is convenient to regard these gravitational instantons as compact spaces with boundary, where the boundary recedes to infinity. Depending on the behaviour near their infinities, the known gravitational instantons fall into four types: asymptotically (locally) Euclidean or asymptotically (locally) flat [113]. A gravitational instanton is said to be asymptotically locally Euclidean (ALE) if near infinity it is diffeomorphic to $\mathbb{R} \times (S^3/\Gamma)$ where Γ is a nontrivial discrete subgroup of $SO(4)$ with free action on S^3, and if the metric tends to the standard flat metric at least as fast as (proper distance)$^{-2}$. In the trivial case when Γ is the identity, the gravitational instanton is said to be asymptotically Euclidean (AE). If near infinity, the metric instead tends to $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2(\sigma_1^2 + \sigma_2^2) + \sigma_3^2$ at least as fast as (proper distance)$^{-1}$, where $\{\sigma_1, \sigma_2, \sigma_3\}$ are the left-invariant one-forms on S^3, the gravitational instanton is said to be asymptotically locally flat (ALF). If near infinity a gravitational instanton is diffeomorphic to $\mathbb{R} \times \mathbb{R} \times S^2$ with identifications made on one of the $\mathbb{R}$'s (along with a possible translation along the azimuthal coordinate of S^2), and if the metric tends to a standard flat metric at least as fast as (proper distance)$^{-1}$, the gravitational instanton is said to be asymptotically flat (AF). We should note, however, that an AF gravitational instanton is not an ALF gravitational instanton with $\Gamma = 1$.

The topology of infinity can readily be read off from the rod structure. First of all, we note that there is an induced $U(1) \times U(1)$ isometry group action on the boundary surface at infinity. In the orbit space, infinity is then represented by a curve far away from any of the finite rods, intersecting only with the first and last (semi-infinite) rods. Suppose that the normalized directions of these two rods are $\ell_1 = (p_1, q_1)$ and $\ell_{N+1} = (p_2, q_2)$ (expressed in the basis (e_1, e_2) consisting of the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry), the topology of infinity is then a lens space $L(q_1p_2 - p_1q_2, w_1q_1 - w_2p_1)$, where the integers w_1, w_2 solve the equation $w_1q_2 - w_2p_2 = \pm 1$ [38]. As we have shown, we can always take $\{e_1 = \ell_2, e_2 = \ell_1\}$. If $\ell_{N+1} = a\ell_1 + b\ell_2$, we have $(p_1, q_1) = (0, 1)$ and $(p_2, q_2) = (b, a)$. The topology of infinity is then $L(b, w_1)$, where w_1 solves the equation $aw_1 - bw_2 = \pm 1$. Since we have now $aw_1 = \pm 1 (\text{mod } b)$, the topology of infinity is $L(b, w_1) \cong L(b, a)$.

Notice that in the case of an AF gravitational instanton (with $U(1) \times U(1)$ isometry), its infinity will have topology $L(0, 1) \cong S^1 \times S^2$. The infinity of an AE gravitational instanton will have topology $L(1, 0) \cong S^3$. In the case where Γ is the cyclic group $\mathbb{Z}_p$, the infinity of an ALE gravitational instanton will have a general lens-space topology $L(p, q)$, with $p \geq 2$; on the other hand, the infinity of an ALF gravitational instanton can have either topology $L(1, 0)$, or $L(p, q)$ with $p \geq 2$.

5.3 Rod structures of known gravitational instantons

In this section, we analyze the rod structures of known gravitational instantons with a $U(1) \times U(1)$ isometry group. We first try to adopt their metrics in the most commonly used form, written in the coordinate system (ψ, ϕ, r, θ). All these metrics are independent of the two coordinates (ψ, ϕ). For each gravitational instanton, we then take the two linearly independent and commuting Killing vector fields as

$\{V_{(1)} = \frac{\partial}{\partial\psi}, V_{(2)} = \frac{\partial}{\partial\phi}\}$, and define the corresponding Weyl–Papapetrou coordinates $(x^1 = \psi, x^2 = \phi, \rho, z)$. The rod structure of the gravitational instanton is then analyzed. As mentioned above, all the rods are space-like, and their directions are written in the form (a_1, a_2) for simplicity, which is, in fact, $a_1\frac{\partial}{\partial\psi} + a_2\frac{\partial}{\partial\phi}$. The pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group of the gravitational instanton is then identified. We also introduce new Weyl–Papapetrou coordinates $(\tilde{x}^1 = \tilde{\psi}, \tilde{x}^2 = \tilde{\phi}, \tilde{\rho}, \tilde{z})$, in which the rod structure of the gravitational instanton has standard orientation. Then the topology of constant r surfaces is studied, and we also point out, in several cases, that the rod structure alone cannot uniquely determine a solution.

5.3.1 Four-dimensional flat space

Four-dimensional flat (Euclidean) space has the well-known metric

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\psi^2 + \cos^2\theta\,\mathrm{d}\phi^2)\,, \tag{5.1}$$

where r and θ take the ranges $r \geq 0$ and $0 \leq \theta \leq \frac{\pi}{2}$.

The Weyl–Papapetrou coordinates (ψ, ϕ, ρ, z) are related to the above coordinates by

$$\rho = \frac{1}{2}\,r^2\sin 2\theta\,, \qquad z = \frac{1}{2}\,r^2\cos 2\theta\,. \tag{5.2}$$

In these coordinates, the rod structure has just a single turning point, at $(\rho = 0, z = 0)$ or $(r = 0)$. It divides the z-axis into two rods:

- Rod 1: a semi-infinite rod located at $(\rho = 0, z \leq 0)$ or $(r \geq 0, \theta = \frac{\pi}{2})$, with (normalized) direction $\ell_1 = (0, 1)$.

- Rod 2: a semi-infinite rod located at $(\rho = 0, z \geq 0)$ or $(r \geq 0, \theta = 0)$, with (normalized) direction $\ell_2 = (1, 0)$.

This rod structure is illustrated in Fig. 5.1.

In this rod structure, the two semi-infinite rods 1 and 2 intersect at the origin. So the orbits generated by $\{\ell_1, \ell_2\}$ should be identified with period 2π independently to ensure regularity, i.e.,

$$(\psi, \phi) \to (\psi, \phi + 2\pi)\,, \qquad (\psi, \phi) \to (\psi + 2\pi, \phi)\,. \tag{5.3}$$

Thus the direction pair $\{\ell_1, \ell_2\}$ is identified as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group of four-dimensional flat space.

Surfaces of constant r carry a naturally induced $U(1) \times U(1)$ isometry group. They are represented by constant r curves in the (ρ, z) half-plane of the orbit space. The analysis of their topology simply follows from the analysis done in section 5.2 for the topology of infinity of a gravitational instanton. It is then obvious that surfaces of constant $r > 0$ in this case have topology S^3; they shrink down to a single point at $r = 0$. It turns out that four-dimensional flat space is the unique AE gravitational instanton [114].

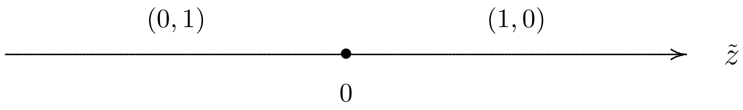

Figure 5.1: The rod structure of four-dimensional flat space and the self-dual Taub-NUT instanton in standard orientation.

5.3.2 Euclidean self-dual Taub-NUT instanton

The Euclidean self-dual Taub-NUT instanton [100, 101] has the metric

$$\mathrm{d}s^2 = H^{-1}(\mathrm{d}\psi + 2n\cos\theta\,\mathrm{d}\phi)^2 + H(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{5.4}$$

where H is a harmonic function on E^3 defined as $H = 1 + \frac{2|n|}{r}$. The parameter n and coordinates r, θ take the ranges $-\infty < n < \infty$, $r \geq 0$, $0 \leq \theta \leq \pi$.

The Weyl–Papapetrou coordinates (ψ, ϕ, ρ, z) are related to the above coordinates by

$$\rho = r\sin\theta\,, \qquad z = r\cos\theta\,. \tag{5.5}$$

In these coordinates, the rod structure has a single turning point at $(\rho = 0, z = 0)$ or $(r = 0)$, just as that of four-dimensional flat space. It consists of the following two rods:

- Rod 1: a semi-infinite rod located at $(\rho = 0, z \leq 0)$ or $(r \geq 0, \theta = \pi)$, with direction $\ell_1 = (2n, 1)$.
- Rod 2: a semi-infinite rod located at $(\rho = 0, z \geq 0)$ or $(r \geq 0, \theta = 0)$, with direction $\ell_2 = (-2n, 1)$.

In this rod structure, the two semi-infinite rods 1 and 2 intersect at the single turning point. So the orbits generated by $\{\ell_1, \ell_2\}$ should be identified with period 2π independently to ensure regularity, i.e.,

$$(\psi, \phi) \to (\psi + 4n\pi, \phi + 2\pi)\,, \qquad (\psi, \phi) \to (\psi - 4n\pi, \phi + 2\pi)\,. \tag{5.6}$$

The direction pair $\{\ell_1, \ell_2\}$ is then identified as the pair of independent 2π-periodic generators of the $U(1)\times U(1)$ isometry group of the self-dual Taub-NUT instanton.

Now, it will prove to be convenient to define a new Killing vector field $\ell_3 = \ell_1 - \ell_2$. Then $\{\ell_1, \ell_3\}$ is related to $\{\ell_1, \ell_2\}$ by a $GL(2, \mathbb{Z})$ transformation, so $\{\ell_1, \ell_3\}$ can also be taken as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group. Since $\ell_3 = (4n, 0)$, we thus have another different but equivalent version of the identifications that can give a regular self-dual Taub-NUT instanton:

$$(\psi, \phi) \to (\psi + 4n\pi, \phi + 2\pi)\,, \qquad (\psi, \phi) \to (\psi + 8n\pi, \phi)\,. \tag{5.7}$$

To make contact with other commonly used forms of this gravitational instanton, define a new coordinate $\psi_S = \psi - 2n\phi$ [79]. Then the metric (5.4) takes the form:

$$\mathrm{d}s^2 = H^{-1}[\mathrm{d}\psi_S + 2n(1 + \cos\theta)\,\mathrm{d}\phi]^2 + H(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{5.8}$$

The corresponding Weyl–Papapetrou coordinates are (ψ_S, ϕ, ρ, z); by changing to these coordinates, the directions of the two semi-infinite rods are now $\ell_1' = (0, 1)$ and $\ell_2' = (-4n, 1)$ respectively (in the basis $(\frac{\partial}{\partial\psi_S}, \frac{\partial}{\partial\phi})$). Then we can take $\{\ell_1', \ell_1' - \ell_2'\}$ as the pair of independent 2π-periodic generators of the $U(1)\times U(1)$ isometry group of the self-dual Taub-NUT instanton, as they are related to $\{\ell_1', \ell_2'\}$ by a $GL(2, \mathbb{Z})$ transformation. Since $\ell_1' - \ell_2' = (4n, 0)$, the coordinates ψ_S and ϕ in the metric

(5.8) should be identified with periods $8n\pi$ and 2π independently, i.e.,

$$(\psi_S, \phi) \to (\psi_S, \phi + 2\pi)\,, \qquad (\psi_S, \phi) \to (\psi_S + 8n\pi, \phi)\,. \tag{5.9}$$

Alternatively, we may define $\psi_N = \psi + 2n\phi$ and show that the same identifications as (5.9) (with ψ_S changed to ψ_N) should be made. Indeed, the two coordinate systems $(\psi_S, \phi, r, \theta)$ and $(\psi_N, \phi, r, \theta)$ were first used by Misner [79] to remove the so-called Dirac–Misner string singularities along the south ($\theta = \pi$) and north pole ($\theta = 0$) of the gravitational instanton, respectively. To ensure that the two coordinate systems are glued together in a compatible way in their overlapping region, Misner inferred the identifications (5.9) and similarly for ψ_N. Here, we have obtained the same identifications by ostensibly different arguments.

We emphasize that the metric (5.4) together with identifications (5.7), or the metric (5.8) together with identifications (5.9), can give the regular self-dual Taub-NUT instanton. But we note that in the literature identifications such as (5.7) or (5.9) are sometimes misused. Taking for example, the metric (5.4) with the identifications (5.9) (with ψ_S changed to ψ), would result in a space with conical singularities.[4] In this case, ϕ cannot be a periodic coordinate with period 2π, unless when accompanied by a translation in ψ.

It is possible to find a new set of coordinates in which the two rod directions have

[4]Identifying ψ and ϕ in the metric (5.4) with periods $8n\pi$ and 2π independently is equivalent to identifying $(4n\frac{\partial}{\partial\psi}, \frac{\partial}{\partial\phi})$ as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group. In the basis $(4n\frac{\partial}{\partial\psi}, \frac{\partial}{\partial\phi})$, we have $\ell_1 = (\frac{1}{2}, 1)$, and $\ell_2 = (-\frac{1}{2}, 1)$. Since the components of ℓ_1 and ℓ_2 are now not integer-valued, there are conical singularities along the corresponding two semi-infinite rods. Similar observations have been made in the past by Feinblum [115].

very simple forms. If $n \neq 0$, we have $\ell_1 \neq \ell_2$. Then we simply take $\{\tilde{V}_{(1)} = \ell_2, \tilde{V}_{(2)} = \ell_1\}$ and define the corresponding new Weyl–Papapetrou coordinates $(\tilde{x}^1 = \tilde{\psi}, \tilde{x}^2 = \tilde{\phi}, \tilde{\rho}, \tilde{z})$, such that $\tilde{V}_i = \frac{\partial}{\partial \tilde{x}^i}$. It is easy to show they are related to the old coordinates (5.5) by

$$\psi = -2n(\tilde{\psi} - \tilde{\phi})\,, \qquad \phi = \tilde{\psi} + \tilde{\phi}\,, \qquad \rho = \frac{1}{4|n|}\tilde{\rho}\,, \qquad z = \frac{1}{4|n|}\tilde{z}\,. \tag{5.10}$$

In these coordinates, the single turning point is located at $(\tilde{\rho} = 0, \tilde{z} = 0)$, and the directions of the two semi-infinite rods are simply $K_1 = (0, 1)$ and $K_2 = (1, 0)$. This is illustrated in Fig. 5.1. We say that the rod structure of the self-dual Taub-NUT instanton has standard orientation in the new Weyl–Papapetrou coordinates $(\tilde{\psi}, \tilde{\phi}, \tilde{\rho}, \tilde{z})$. The direction pair $\{K_1, K_2\}$ is then identified as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group. So in the new Weyl–Papapetrou coordinates, instead of (5.6), the following identifications should be made to ensure regularity:

$$(\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi}, \tilde{\phi} + 2\pi)\,, \qquad (\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi} + 2\pi, \tilde{\phi})\,. \tag{5.11}$$

Now we can see that, in appropriately chosen coordinates, the self-dual Taub-NUT instanton with nonzero NUT charge n has exactly the same rod structure as four-dimensional flat space. Furthermore, the self-dual Taub-NUT instanton with different NUT charges n all share the same rod structure. This is an example of the fact that the rod structure alone cannot uniquely determine a solution.

From the rod structure in standard orientation, we can easily see that, just as in the case of four-dimensional flat space, surfaces of constant $r > 0$ have topology S^3, and they shrink down to a point at $r = 0$. At infinity $r \to \infty$, the self-dual Taub-NUT instanton approaches a finite S^1 fibre bundle over an S^2. This is nothing but the well-known Hopf fibration of S^3. The Killing vector field $\frac{\partial}{\partial\psi}$ generates the finite S^1 fibre at infinity, with a constant size $8n\pi$. It can be checked that the self-dual Taub-NUT instanton is ALF with $\Gamma = 1$. When $n \to 0$, the S^1 dimension vanishes,[5] and we thus recover a three-dimensional flat space; on the other hand, when $n \to \infty$, the S^1 dimension blows up, and we recover a four-dimensional flat space.

We note that the above non-uniqueness result does not violate the uniqueness theorem proved by Hollands and Yazadjiev [39], as the self-dual Taub-NUT instanton is not within the class of spatial backgrounds of the solutions considered in [39]. Recall that the asymptotic geometry of the space-times considered in [39] is $M^{1,s} \times T^{D-s-1}$, with an $\mathbb{R} \times U(1)^{D-3}$ isometry group. The latter generates translations along time, and the standard rotations in the Minkowski space-time $M^{1,s}$ and the flat torus T^{D-s-1}. On the other hand, the self-dual Taub-NUT instanton approaches a non-trivial S^1 fibre bundle over S^2 at infinity, with a crucial non-vanishing $g_{\psi\phi}$ cross term. In this case, the pair of independent 2π-periodic generators $\{\ell_1, \ell_2\}$ of the $U(1) \times U(1)$ isometry group generates rotations along the south and north poles respectively of a distorted S^3.

5.3.3 Euclidean Schwarzschild instanton

The Euclidean Schwarzschild instanton has the metric

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right) \mathrm{d}\psi^2 + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2)\,, \tag{5.12}$$

[5]This can be seen from the metric (5.4), together with the identifications (5.7). ψ has a period which vanishes when $n \to 0$, so the S^1 dimension shrinks down to zero.

where the parameter m and coordinates r, θ take the ranges $r \geq 2m \geq 0$, $0 \leq \theta \leq \pi$.

The Weyl–Papapetrou coordinates (ψ, ϕ, ρ, z) are related to the above coordinates by

$$\rho = \sqrt{r^2 - 2mr}\,\sin\theta\,, \qquad z = (r-m)\cos\theta\,. \tag{5.13}$$

In these coordinates, the rod structure has two turning points at $(\rho = 0, z = z_1 \equiv -m)$ or $(r = 2m, \theta = \pi)$, and at $(\rho = 0, z = z_2 \equiv m)$ or $(r = 2m, \theta = 0)$. They divide the z-axis into three rods; from left to right they are:

- Rod 1: a semi-infinite rod located at $(\rho = 0, z \leq z_1)$ or $(r \geq 2m, \theta = \pi)$, with direction $\ell_1 = (0, 1)$.
- Rod 2: a finite rod located at $(\rho = 0, z_1 \leq z \leq z_2)$ or $(r = 2m, 0 \leq \theta \leq \pi)$, with direction $\ell_2 = (4m, 0)$.
- Rod 3: a semi-infinite rod located at $(\rho = 0, z \geq z_2)$ or $(r \geq 2m, \theta = 0)$, with direction $\ell_3 = (0, 1)$.

It is straightforward to check that the direction pairs $\{\ell_1, \ell_2\}$ and $\{\ell_2, \ell_3\}$ of adjacent rods are related by a $GL(2, \mathbb{Z})$ transformation. To ensure regularity, the orbits generated by say the first direction pair $\{\ell_1, \ell_2\}$ should be identified with period 2π independently, i.e.,

$$(\psi, \phi) \to (\psi, \phi + 2\pi)\,, \qquad (\psi, \phi) \to (\psi + 8m\pi, \phi)\,. \tag{5.14}$$

The direction pair $\{\ell_1, \ell_2\}$, or equivalently $\{\ell_2, \ell_3\}$, is then identified as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group of the Euclidean Schwarzschild instanton.

If $m \neq 0$, we can put the rod structure in standard orientation by taking $\{\tilde{V}_{(1)} = \ell_2, \tilde{V}_{(2)} = \ell_1\}$. The corresponding new Weyl–Papapetrou coordinates $(\tilde{\psi}, \tilde{\phi}, \tilde{\rho}, \tilde{z})$ are related to the old coordinates (5.13) by

$$\psi = 4m\tilde{\psi}\,, \qquad \phi = \tilde{\phi}\,, \qquad \rho = \frac{1}{4m}\tilde{\rho}\,, \qquad z = \frac{1}{4m}\tilde{z}\,. \tag{5.15}$$

The two turning points are now pushed to $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_1 \equiv -4m^2)$ and $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_2 \equiv 4m^2)$, and the corresponding directions of the three rods from left to right are $K_1 = (0, 1)$, $K_2 = (1, 0)$ and $K_3 = (0, 1)$. This is illustrated in Fig. 5.2(a). It is clear that in the new Weyl–Papapetrou coordinates, the following identifications should be made to ensure regularity:

$$(\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi}, \tilde{\phi} + 2\pi)\,, \qquad (\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi} + 2\pi, \tilde{\phi})\,. \tag{5.16}$$

The topology of the constant $r > r_0$ surfaces is $S^1 \times S^2$, with ℓ_2 and ℓ_1 being the generators of the rotations for S^1 and S^2 respectively. When $r \to 2m$, the S^1 vanishes, and the constant r surface becomes a two-sphere S^2 at $r = 2m$. At infinity $r \to \infty$, the S^1 dimension approaches a constant size $8m\pi$. It can be checked that the Euclidean Schwarzschild instanton is AF. When $m \to 0$, the S^1 dimension vanishes, and we recover a three-dimensional flat space.

As a final remark, note that if we take the limit $m \to 0$ without imposing the second identification of (5.14), we obtain a $U(1) \times \mathbb{R}^3$ space with flat metric, in which the ψ coordinate parameterizing the $U(1)$ can have any period. It is the

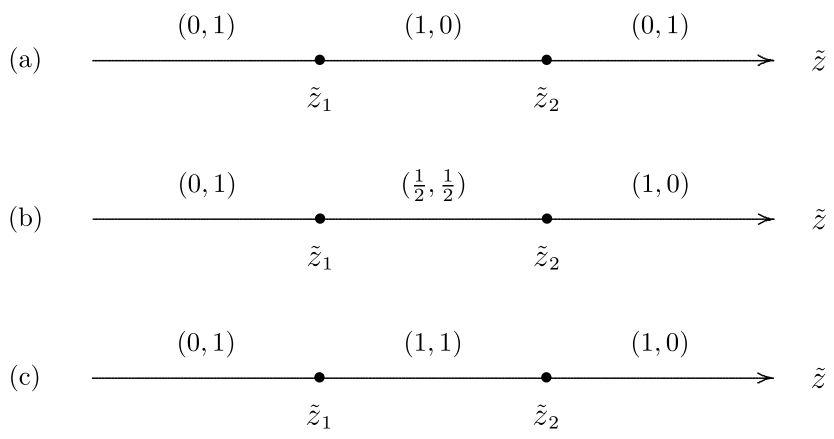

Figure 5.2: The rod structure of: (a) the Euclidean Schwarzschild and Kerr instantons; (b) the Eguchi–Hanson and double-centered Taub-NUT instantons; and (c) the Taub-bolt instanton; all in standard orientation.

trivial AF gravitational instanton, whose rod structure consists of a single infinite rod with direction (0,1).

5.3.4 Euclidean Kerr instanton

The Euclidean Kerr instanton [104] has the metric

$$\mathrm{d}s^2 = \frac{\Delta\left(\mathrm{d}\psi + a\sin^2\theta\,\mathrm{d}\phi\right)^2}{\Sigma} + \frac{\sin^2\theta\,[a\mathrm{d}\psi - (r^2 - a^2)\,\mathrm{d}\phi]^2}{\Sigma} + \Sigma\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right), \tag{5.17}$$

where Σ and Δ are defined as

$$\Sigma = r^2 - a^2\cos^2\theta\,, \qquad \Delta = r^2 - 2mr - a^2\,. \tag{5.18}$$

The parameters m, a and coordinates r, θ take the ranges $m \geq 0$, $-\infty < a < \infty$, $r \geq r_0$, $0 \leq \theta \leq \pi$, where r_0 is defined as the larger root of Δ, i.e., $r_0 = m +$

$\sqrt{m^2 + a^2}$.

The Weyl–Papapetrou coordinates (ψ, ϕ, ρ, z) are related to the above coordinates by

$$\rho = \sqrt{r^2 - 2mr - a^2} \sin\theta \,, \qquad z = (r - m)\cos\theta \,. \tag{5.19}$$

In these coordinates, the rod structure is similar to that of the Euclidean Schwarzschild instanton. There are two turning points, located at $(\rho = 0, z = z_1 \equiv -\sqrt{m^2 + a^2})$ or $(r = r_0, \theta = \pi)$, and $(\rho = 0, z = z_2 \equiv \sqrt{m^2 + a^2})$ or $(r = r_0, \theta = 0)$. They divide the z-axis into three rods; from left to right they are:

- Rod 1: a semi-infinite rod located at $(\rho = 0, z \le z_1)$ or $(r \ge r_0, \theta = \pi)$, with direction $\ell_1 = (0, 1)$.
- Rod 2: a finite rod located at $(\rho = 0, z_1 \le z \le z_2)$ or $(r = r_0, 0 \le \theta \le \pi)$, with direction $\ell_2 = (\frac{1}{\kappa_E}, \frac{\Omega_E}{\kappa_E})$, where κ_E is the Euclidean surface gravity on the horizon and Ω_E is the Euclidean angular velocity of the horizon, given by

$$\kappa_E = \frac{\sqrt{m^2 + a^2}}{2m(m + \sqrt{m^2 + a^2})} \,, \qquad \Omega_E = \frac{a}{2m(m + \sqrt{m^2 + a^2})} \,. \tag{5.20}$$

- Rod 3: a semi-infinite rod located at $(\rho = 0, z \ge z_2)$ or $(r \ge r_0, \theta = 0)$, with direction $\ell_3 = (0, 1)$.

It is straightforward to check that the direction pairs $\{\ell_1, \ell_2\}$ and $\{\ell_2, \ell_3\}$ of adjacent rods are related by a $GL(2, \mathbb{Z})$ transformation. To ensure regularity, the orbits generated by say the first direction pair $\{\ell_1, \ell_2\}$ should be identified with period 2π independently, i.e.,

$$(\psi, \phi) \to (\psi, \phi + 2\pi)\,, \qquad (\psi, \phi) \to \left(\psi + \frac{2\pi}{\kappa_E}, \phi + \frac{2\pi\Omega_E}{\kappa_E}\right)\,. \tag{5.21}$$

The direction pair $\{\ell_1, \ell_2\}$, or equivalently $\{\ell_2, \ell_3\}$, is then identified as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group of the Euclidean Kerr instanton.

Obviously, when $a \to 0$ we recover the Euclidean Schwarzschild instanton from the Euclidean Kerr instanton. On the other hand, when $m \to 0$ we recover a three-dimensional flat space in a new form; this is similar to the $m \to 0$ limit of the Euclidean Schwarzschild instanton.

If $m \neq 0$, we can put the rod structure in standard orientation by taking $\{\tilde{V}_{(1)} = \ell_2, \tilde{V}_{(2)} = \ell_1\}$. The corresponding new Weyl–Papapetrou coordinates $(\tilde{\psi}, \tilde{\phi}, \tilde{\rho}, \tilde{z})$ are related to the old coordinates (5.19) by

$$\psi = \frac{1}{\kappa_E}\,\tilde{\psi}\,, \qquad \phi = \frac{\Omega_E}{\kappa_E}\,\tilde{\psi} + \tilde{\phi}\,, \qquad \rho = \kappa_E \tilde{\rho}\,, \qquad z = \kappa_E \tilde{z}\,. \tag{5.22}$$

The two turning points are now pushed to $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_1 \equiv -2m(m + \sqrt{m^2 + a^2}))$ and $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_2 \equiv 2m(m + \sqrt{m^2 + a^2}))$, and the corresponding directions of the three rods from left to right are $K_1 = (0, 1)$, $K_2 = (1, 0)$ and $K_3 = (0, 1)$. This is illustrated in Fig. 5.2(a). In the new Weyl–Papapetrou coordinates, the following identifications should be made to ensure regularity:

$$(\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi}, \tilde{\phi} + 2\pi)\,, \qquad (\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi} + 2\pi, \tilde{\phi})\,. \tag{5.23}$$

As in the case of the Euclidean Schwarzschild instanton, the topology of the constant $r > r_0$ surfaces is $S^1 \times S^2$, with ℓ_2 and ℓ_1 being the generators of the rotations for S^1 and S^2 respectively. When $r \to r_0$, the S^1 vanishes, and the constant r surface becomes a two-sphere S^2 at $r = r_0$. But unlike the situation in the Euclidean Schwarzschild instanton, the S^1 blows up at infinity $r \to \infty$ in the general case for $a \neq 0$ [116], which can be seen from the fact that $g_{\tilde{\psi}\tilde{\psi}}$ diverges at infinity. However, in this general case, if $-1 < \frac{\Omega_E}{\kappa_E} < 1$ is a rational number $\frac{q}{p}$ with coprime integers p and q, then the Killing vector field $\frac{\partial}{\partial\psi}$ generates closed and finite orbits with period $\frac{2\pi p}{\kappa_E}$ at infinity [117]. It can be checked that, like the Euclidean Schwarzschild instanton, the Euclidean Kerr instanton is AF.

The rod structure of the Euclidean Kerr instanton admits a one-parameter degeneracy, which can be seen from the fact that we can appropriately vary m and a such that the turning points $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_1)$ and $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_2)$ remain fixed. We also note that there is a one-parameter family of the Euclidean Kerr instanton that has exactly the same rod structure (in standard orientation) as the Euclidean Schwarzschild instanton. This is another example of the fact that the rod structure alone cannot uniquely determine a solution, even when NUT charge is absent. And again this does not violate Hollands and Yazadjiev's theorem [39], as the Euclidean Kerr instanton has an S^1 circle whose radius diverges at infinity, and moreover its metric has a cross term $g_{\tilde{\psi}\tilde{\phi}}$ that diverges at infinity.

5.3.5 Eguchi–Hanson instanton

The Eguchi–Hanson instanton [105] has the well-known metric

$$\mathrm{d}s^2 = \left(1 - \frac{a^4}{r^4}\right)\frac{r^2}{4}\left(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\right)^2 + \left(1 - \frac{a^4}{r^4}\right)^{-1}\mathrm{d}r^2 + \frac{r^2}{4}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{5.24}$$

where the parameter a and coordinates r, θ take the ranges $r \geq a > 0$, $0 \leq \theta \leq \pi$.

The Weyl–Papapetrou coordinates (ψ, ϕ, ρ, z) are related to the above coordinates by

$$\rho = \frac{1}{4}\sqrt{r^4 - a^4}\,\sin\theta\,, \qquad z = \frac{r^2}{4}\cos\theta\,. \tag{5.25}$$

In these coordinates, the rod structure has two turning points at $(\rho = 0, z = z_1 \equiv -\frac{a^2}{4})$ or $(r = a, \theta = \pi)$, and at $(\rho = 0, z = z_2 \equiv \frac{a^2}{4})$ or $(r = a, \theta = 0)$. It consists of the following three rods:

- Rod 1: a semi-infinite rod located at $(\rho = 0, z \leq z_1)$ or $(r \geq a, \theta = \pi)$, with direction $\ell_1 = (1, 1)$.
- Rod 2: a finite rod located at $(\rho = 0, z_1 \leq z \leq z_2)$ or $(r = a, 0 \leq \theta \leq \pi)$, with direction $\ell_2 = (1, 0)$.
- Rod 3: a semi-infinite rod located at $(\rho = 0, z \geq z_2)$ or $(r \geq a, \theta = 0)$, with direction $\ell_3 = (-1, 1)$.

It is straightforward to check that the direction pairs $\{\ell_1, \ell_2\}$ and $\{\ell_2, \ell_3\}$ of adjacent rods are related by a $GL(2, \mathbb{Z})$ transformation. To ensure regularity, the orbits generated by say the first direction pair $\{\ell_1, \ell_2\}$ should be identified with period 2π independently. This direction pair can then be taken as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group of the Eguchi–Hanson instanton. However, since the pair $\{\ell_1 - \ell_2, \ell_2\}$ is also related to $\{\ell_1, \ell_2\}$ by a $GL(2, \mathbb{Z})$ transformation, and since $\ell_1 - \ell_2 = (0, 1)$, we can make the following identifications to ensure regularity:

$$(\psi, \phi) \to (\psi, \phi + 2\pi)\,, \qquad (\psi, \phi) \to (\psi + 2\pi, \phi)\,. \tag{5.26}$$

Note that in this case, the direction pair $\{\ell_1, \ell_3\}$ corresponding to the two semi-infinite rods is not related to $\{\ell_1, \ell_2\}$ by a $GL(2, \mathbb{Z})$ transformation, so it cannot be taken as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group.

We can put the rod structure in standard orientation by taking $\{\tilde{V}_{(1)} = -\ell_3, \tilde{V}_{(2)} = \ell_1\}$. The corresponding new Weyl–Papapetrou coordinates $(\tilde{\psi}, \tilde{\phi}, \tilde{\rho}, \tilde{z})$ are related to the old coordinates (5.25) by

$$\psi = \tilde{\psi} + \tilde{\phi}\,, \qquad \phi = -\tilde{\psi} + \tilde{\phi}\,, \qquad \rho = \frac{1}{2}\tilde{\rho}\,, \qquad z = \frac{1}{2}\tilde{z}\,. \tag{5.27}$$

The two turning points are now pushed to $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_1 \equiv -\frac{a^2}{2})$ and $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_2 \equiv \frac{a^2}{2})$, and the corresponding directions of the three rods from left to right are $K_1 = (0, 1)$, $K_2 = (\frac{1}{2}, \frac{1}{2})$ and $K_3 = (1, 0)$.[6] This is illustrated in Fig. 5.2(b). In the new Weyl–Papapetrou coordinates, the following identifications should be made to ensure regularity:

$$(\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi}, \tilde{\phi} + 2\pi)\,, \qquad (\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi} + \pi, \tilde{\phi} + \pi)\,. \tag{5.28}$$

Surfaces of constant $r > a$ are represented in the (ρ, z) half-plane of the orbit space by curves intersecting with rod 1 and 3. Since $K_3 = 2K_2 - K_1$, the topology of these surfaces is then a lens-space $L(2, -1) \cong \mathbb{R}P^3$. They shrink down to a two-sphere at $r = a$. At infinity $r \to \infty$, the Eguchi–Hanson instanton approaches a four-dimensional flat space quotiented by a $\mathbb{Z}_2$ group, i.e., $E^4/\mathbb{Z}_2$. It can be checked that this instanton is ALE with $\Gamma = \mathbb{Z}_2$.

[6] Strictly speaking, we should have $K_3 = (-1, 0)$ to keep the direction of rod 3 invariant under the coordinate transformation (5.27). But recall that we allow a possible minus sign for the direction of a rod, so here we take $K_3 = (1, 0)$.

Finally, we note that, under the above identifications, the limit $a \to 0$ results in a four-dimensional flat space quotiented by a $\mathbb{Z}_2$ group, which is singular as there is an orbifold singularity present at the origin.

5.3.6 Double-centered Taub-NUT instanton

The double-centered Taub-NUT instanton [101, 108] has the metric

$$\mathrm{d}s^2 = H^{-1}(\mathrm{d}\psi + A)^2 + H(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{5.29}$$

where the harmonic function H and the twist potential A on E^3 are defined as

$$\begin{aligned} H &= 1 + \frac{2|n_1|}{r_1} + \frac{2|n_2|}{r_2}\,, \\ A &= \frac{2n_1\,(r\cos\theta + a)}{r_1}\,\mathrm{d}\phi + \frac{2n_2\,(r\cos\theta - a)}{r_2}\,\mathrm{d}\phi\,, \end{aligned} \tag{5.30}$$

with $r_1 = \sqrt{r^2 + a^2 + 2ar\cos\theta}$ and $r_2 = \sqrt{r^2 + a^2 - 2ar\cos\theta}$. The parameters a, n_1, n_2 and coordinates r, θ take the ranges $a > 0$, $-\infty < n_1, n_2 < \infty$, $r \geq 0$, $0 \leq \theta \leq \pi$. Furthermore, the NUT charges n_1 and n_2 are required to have the same sign.

The Weyl–Papapetrou coordinates (ψ, ϕ, ρ, z) are related to the above coordinates by

$$\rho = r\sin\theta\,, \qquad z = r\cos\theta\,. \tag{5.31}$$

In these coordinates, the rod structure has two turning points at $(\rho = 0, z = z_1 \equiv -a)$ or $(r = a, \theta = \pi)$, and at $(\rho = 0, z = z_2 \equiv a)$ or $(r = a, \theta = 0)$. It consists of the following three rods:

- Rod 1: a semi-infinite rod located at $(\rho = 0, z \leq z_1)$ or $(r \geq a, \theta = \pi)$, with direction $\ell_1 = (2n_1 + 2n_2, 1)$.
- Rod 2: a finite rod located at $(\rho = 0, z_1 \leq z \leq z_2)$ or $(0 \leq r \leq a,\ \theta = 0 \text{ and } \pi)$, with direction $\ell_2 = (-2n_1 + 2n_2, 1)$.
- Rod 3: a semi-infinite rod located at $(\rho = 0, z \geq z_2)$ or $(r \geq a, \theta = 0)$, with direction $\ell_3 = (-2n_1 - 2n_2, 1)$.

It can be checked that the direction pairs $\{\ell_1, \ell_2\}$ and $\{\ell_2, \ell_3\}$ of adjacent rods are related by a $GL(2, \mathbb{Z})$ transformation, provided the two NUT charges are equal: $n_1 = n_2 \equiv n$; this is what we assume from now on. To further ensure regularity, the orbits generated by say the first direction pair $\{\ell_1, \ell_2\}$ should be identified with period 2π independently. This direction pair can then be taken as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group of the double-centered Taub-NUT instanton. However, since the pair $\{\ell_1 - \ell_2, \ell_2\}$ is also related to $\{\ell_1, \ell_2\}$ by a $GL(2, \mathbb{Z})$ transformation, and since $\ell_1 - \ell_2 = (4n, 0)$, we can make the following identifications to ensure regularity:

$$(\psi, \phi) \to (\psi + 8n\pi, \phi)\,, \qquad (\psi, \phi) \to (\psi, \phi + 2\pi)\,. \tag{5.32}$$

As in the Eguchi–Hanson instanton, the direction pair $\{\ell_1, \ell_3\}$ corresponding to the two semi-infinite rods is not related to $\{\ell_1, \ell_2\}$ by a $GL(2, \mathbb{Z})$ transformation, so it cannot be taken as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group.

If $n \neq 0$, we can put the rod structure in standard orientation by taking $\{\tilde{V}_{(1)} = \ell_3, \tilde{V}_{(2)} = \ell_1\}$. The corresponding new Weyl–Papapetrou coordinates $(\tilde{\psi}, \tilde{\phi}, \tilde{\rho}, \tilde{z})$

are related to the old coordinates (5.31) by

$$\psi = -4n(\tilde{\psi} - \tilde{\phi}) , \qquad \phi = \tilde{\psi} + \tilde{\phi} , \qquad \rho = \frac{1}{8|n|}\tilde{\rho} , \qquad z = \frac{1}{8|n|}\tilde{z} . \tag{5.33}$$

The two turning points are now pushed to $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_1 \equiv -8|n|a)$ and $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_2 \equiv 8|n|a)$, and the corresponding directions of the three rods from left to right are $K_1 = (0, 1)$, $K_2 = (\frac{1}{2}, \frac{1}{2})$ and $K_3 = (1, 0)$. This is illustrated in Fig. 5.2(b). In the new Weyl–Papapetrou coordinates, the following identifications should be made to ensure regularity:

$$(\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi}, \tilde{\phi} + 2\pi) , \qquad (\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi} + \pi, \tilde{\phi} + \pi) . \tag{5.34}$$

It is easy to show that constant r surfaces have the topology $S^1 \times S^2$ for $0 < r < a$, and the topology $L(2, 1)$ for $r > a$. At infinity $r \to \infty$, the Killing vector field $\frac{\partial}{\partial\psi}$ generates a compact dimension of constant size $8n\pi$. It can be checked that the double-centered Taub-NUT instanton is ALF with $\Gamma = \mathbb{Z}_2$. When $n \to 0$, we recover a three-dimensional flat space; on the other hand, when $n \to \infty$, the finite dimension blows up, and we recover the Eguchi–Hanson instanton.

As can be seen, the above rod structure also admits a one-parameter degeneracy. We also note that there is a one-parameter family of the double-centered Taub-NUT instanton (5.29) that has exactly the same rod structure (in standard orientation) as the Eguchi–Hanson instanton (5.24). This is the third example that the rod structure alone cannot uniquely determine a solution. Actually the double-centered Taub-NUT instanton generalizes the Eguchi–Hanson instanton, in the same way that the self-dual Taub-NUT instanton generalizes four-dimensional flat space. So the same kind of degeneracy of the rod structure appears.

5.3.7 Taub-bolt instanton

The Euclidean non-self-dual Taub-NUT solution has the metric

$$\mathrm{d}s^2 = f(r)\,(\mathrm{d}\psi + 2n\cos\theta\,\mathrm{d}\phi)^2 + \frac{\mathrm{d}r^2}{f(r)} + \left(r^2 - n^2\right)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \qquad (5.35)$$

where the function $f(r)$ is defined as

$$f(r) = \frac{r^2 + n^2 - 2mr}{r^2 - n^2}\,. \qquad (5.36)$$

The parameters m, n and coordinates r, θ take the ranges $m \geq |n|$, $r \geq r_0$, $0 \leq \theta \leq \pi$, where r_0 is the larger root of $f(r)$, i.e., $r_0 = m + \sqrt{m^2 - n^2}$. The self-dual Taub-NUT metric (5.4) can be recovered from (5.35) by setting $m = |n|$.

The Weyl–Papapetrou coordinates (ψ, ϕ, ρ, z) are related to the above coordinates by

$$\rho = \sqrt{r^2 - 2mr + n^2}\,\sin\theta\,, \qquad z = (r - m)\cos\theta\,. \qquad (5.37)$$

In these coordinates, the rod structure has two turning points at $(\rho = 0, z = z_1 \equiv -\sqrt{m^2 - n^2})$ or $(r = r_0, \theta = \pi)$, and at $(\rho = 0, z = z_2 \equiv \sqrt{m^2 - n^2})$ or $(r = r_0, \theta = 0)$. It consists of the following three rods:

- Rod 1: a semi-infinite rod located at $(\rho = 0, z \leq z_1)$ or $(r \geq r_0, \theta = \pi)$, with direction $\ell_1 = (2n, 1)$.
- Rod 2: a finite rod located at $(\rho = 0, z_1 \leq z \leq z_2)$ or $(r = r_0, 0 \leq \theta \leq \pi)$, with direction $\ell_2 = (2m + 2\sqrt{m^2 - n^2}, 0)$.
- Rod 3: a semi-infinite rod located at $(\rho = 0, z \geq z_2)$ or $(r \geq r_0, \theta = 0)$, with direction $\ell_3 = (-2n, 1)$.

It can be checked that the direction pairs $\{\ell_1, \ell_2\}$ and $\{\ell_2, \ell_3\}$ of adjacent rods are related by a $GL(2, \mathbb{Z})$ transformation, provided $m = \frac{5}{4}|n|$. This corresponds to the Taub-bolt instanton discovered by Page [106]. In what follows, we focus on the Taub-bolt instanton only; in this case, we can take $\ell_2 = (4n, 0)$ without loss of generality. To further ensure regularity, the orbits generated by say the first direction pair $\{\ell_1, \ell_2\}$ should be identified with period 2π independently, i.e.,

$$(\psi, \phi) \to (\psi + 4n\pi, \phi + 2\pi)\,, \qquad (\psi, \phi) \to (\psi + 8n\pi, \phi)\,. \tag{5.38}$$

These identifications are, in fact, the same as (5.7) needed to make the self-dual Taub-NUT instanton (5.4) regular. This is because the direction pair $\{\ell_1, \ell_3\}$ corresponding to the two semi-infinite rods is related to the direction pair $\{\ell_1, \ell_2\}$ by a $GL(2, \mathbb{Z})$ transformation, and so can be taken as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group of the Taub-bolt instanton.

If $n \neq 0$, we can put the rod structure in standard orientation by taking $\{\tilde{V}_{(1)} = -\ell_3, \tilde{V}_{(2)} = \ell_1\}$. The corresponding new Weyl–Papapetrou coordinates $(\tilde{\psi}, \tilde{\phi}, \tilde{\rho}, \tilde{z})$ are related to the old coordinates (5.37) by

$$\psi = 2n(\tilde{\psi} + \tilde{\phi})\,, \qquad \phi = -\tilde{\psi} + \tilde{\phi}\,, \qquad \rho = \frac{1}{4|n|}\tilde{\rho}\,, \qquad z = \frac{1}{4|n|}\tilde{z}\,. \tag{5.39}$$

The two turning points are now pushed to $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_1 \equiv -3n^2)$ and $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_2 \equiv 3n^2)$, and the corresponding directions of the three rods from left to right are $K_1 = (0, 1)$, $K_2 = (1, 1)$ and $K_3 = (1, 0)$.[7] This is illustrated in Fig. 5.2(c). Since $\{K_1, K_3\}$ can be taken as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group, in the new Weyl–Papapetrou coordinates the

[7] As in the case of the Eguchi–Hanson instanton, we should strictly speaking have $K_3 = (-1, 0)$, but here we take $K_3 = (1, 0)$ without making any difference.

following identifications can be made to ensure regularity:

$$(\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi}, \tilde{\phi} + 2\pi)\,, \qquad (\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi} + 2\pi, \tilde{\phi})\,. \tag{5.40}$$

Note that the Eguchi–Hanson and double-centered Taub-NUT instantons share the same Harmark rod structure [36, 37] as the Taub-bolt instanton (in appropriate coordinates), but the stronger form of the rod structure as defined in this thesis is able to distinguish the two cases.

It is easy to show that surfaces of constant $r > r_0$ have topology S^3, and that they shrink down to a two-sphere S^2 at $r = r_0$. At infinity $r \to \infty$, the Taub-bolt instanton approaches a finite S^1 fibre bundle over an S^2, which is the Hopf fibration of S^3. The Killing vector field $\frac{\partial}{\partial \psi}$ generates the finite S^1 fibre at infinity, with a constant size $8n\pi$. Like the self-dual Taub-NUT instanton, the Taub-bolt instanton is ALF with $\Gamma = 1$. When $n \to 0$, we recover a three-dimensional flat space.

Finally, we point out that the Eguchi–Hanson metric (5.24) can be recovered from the non-self-dual Taub-NUT metric (5.35) up to coordinate transformations, by taking the limit $m \to \infty$ with $m^4 - n^4$ fixed.

5.3.8 No completely regular Kerr-bolt instanton

The Kerr-bolt instanton was first discussed by Gibbons and Perry [107]. It generalizes the Taub-bolt instanton in the same way that the Kerr solution generalizes the Schwarzschild solution, and was obtained as a special case of the Euclidean Kerr-NUT metric. Here we take a simpler form of the Euclidean Kerr-NUT metric as used in Ghezelbash et al. [109]:

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\Delta}{\Sigma}[\mathrm{d}\psi + \left(2n\cos\theta + a\sin^2\theta\right)\mathrm{d}\phi]^2 + \frac{\sin^2\theta}{\Sigma}[a\mathrm{d}\psi - \left(r^2 - n^2 - a^2\right)\mathrm{d}\phi]^2 \\ &\quad + \Sigma\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right), \end{aligned} \tag{5.41}$$

where Σ and Δ are defined as

$$\Sigma = r^2 - (n - a\cos\theta)^2, \qquad \Delta = r^2 - 2mr - a^2 + n^2. \tag{5.42}$$

The parameters m, n, a and coordinates r, θ take the ranges $m \geq |n|$, $-\infty < a < \infty$, $r \geq r_0$, $0 \leq \theta \leq \pi$, where r_0 is the larger root of Δ, i.e., $r_0 = m + \sqrt{m^2 + a^2 - n^2}$. The Euclidean Kerr metric (5.17) and the non-self-dual Taub-NUT metric (5.35) can be recovered from (5.41) by setting $n = 0$ and $a = 0$, respectively.

The Weyl–Papapetrou coordinates (ψ, ϕ, ρ, z) are related to the above coordinates by

$$\rho = \sqrt{r^2 - 2mr - a^2 + n^2}\,\sin\theta, \qquad z = (r - m)\cos\theta. \tag{5.43}$$

In these coordinates, the rod structure has two turning points, at $(\rho = 0, z = z_1 \equiv -\sqrt{m^2 + a^2 - n^2})$ or $(r = r_0, \theta = \pi)$, and at $(\rho = 0, z = z_2 \equiv \sqrt{m^2 + a^2 - n^2})$ or $(r = r_0, \theta = 0)$. It consists of the following three rods:

- Rod 1: a semi-infinite rod located at $(\rho = 0, z \leq z_1)$ or $(r \geq r_0, \theta = \pi)$, with direction $\ell_1 = (2n, 1)$.
- Rod 2: a finite rod located at $(\rho = 0, z_1 \leq z \leq z_2)$ or $(r = r_0, 0 \leq \theta \leq \pi)$, with direction $\ell_2 = (\frac{1}{\kappa_E}, \frac{\Omega_E}{\kappa_E})$, where κ_E and Ω_E are defined as

$$\kappa_E = \frac{\sqrt{m^2 + a^2 - n^2}}{2(m^2 - n^2 + m\sqrt{m^2 + a^2 - n^2})}, \qquad \Omega_E = \frac{a}{2(m^2 - n^2 + m\sqrt{m^2 + a^2 - n^2})}. \tag{5.44}$$

- Rod 3: a semi-infinite rod located at $(\rho = 0, z \geq z_2)$ or $(r \geq r_0, \theta = 0)$, with direction $\ell_3 = (-2n, 1)$.

Without loss of generality, we consider the case $0 \leq n \leq m$; a similar analysis applies when $-m \leq n \leq 0$. When $n = 0$, we recover the Euclidean Kerr instanton, in which case $\ell_1 = \ell_3$. When $n = m$, we have $\ell_1 = \ell_2$ and thus in this case the first and the second rods actually join up to form a single rod, leaving only one turning point in the rod structure. The resulting metric is nothing but the self-dual Taub-NUT metric (5.4) in a different form.

When $0 < n < m$, all three directions are mutually linearly independent. If we impose the condition that the direction pairs $\{\ell_1, \ell_2\}$ and $\{\ell_2, \ell_3\}$, of adjacent rods intersecting at the two turning points, are related by a $GL(2, \mathbb{Z})$ transformation, it is straightforward to show that the only solution is $\{a = 0, m = \frac{5n}{4}\}$, which is just the Taub-bolt instanton. This result implies that no new gravitational instantons can be obtained from (5.41) when $a \neq 0$; in particular, there is no Kerr-bolt instanton satisfying our regularity conditions!

It is worth examining this result in more detail, especially in the light of previous analyses of the Kerr-bolt instanton. In order to avoid conical singularities along all three rods, the orbits generated by ℓ_1, ℓ_2 and ℓ_3 should be respectively identified with period 2π; at the same time, it is necessary for the so-called compatibility condition $q\ell_1 + p\ell_2 + s\ell_3 = 0$ to be satisfied for mutually coprime integers p, q and s (see e.g., [109] for an explanation of this and some information on lattice analysis). This class of Kerr-bolt instantons free of conical singularities was analyzed in detail in [109]. However, it turns out that they contain orbifold singularities in general.

We recall that, by first identifying the orbits generated by ℓ_1 and ℓ_2 with period 2π respectively, the metric in the vicinity of the first turning point $(\rho = 0, z = z_1)$ can be brought into the standard form of E^4 near the origin (3.11), with $\ell_1 = \frac{\partial}{\partial \phi_1}$ and $\ell_2 = \frac{\partial}{\partial \phi_2}$. Since $\ell_3 = -\frac{1}{s}(q\ell_1 + p\ell_2)$, if we further identify the orbits generated by ℓ_3 with period 2π, we quotient the space by a $\mathbb{Z}_{|s|}$ identification group. This will leave a $\mathbb{Z}_{|s|}$ orbifold singularity at the first turning point if $|s| \geq 2$. Similarly, if $|q| \geq 2$ there will be a $\mathbb{Z}_{|q|}$ orbifold singularity at the second turning point $(\rho = 0, z = z_2)$.[8] If we require the absence of these two orbifold singularities, we have to impose $s = \pm 1$ and $q = \pm 1$. It is easy to see that the only solution to these conditions is $\{a = 0, m = \frac{5n}{4}\}$, i.e., the Taub-bolt instanton.

Hence, there does not exist a completely regular Kerr-bolt instanton free of both conical and orbifold singularities. The space with metric (5.41) is regular only in two special cases, the Euclidean Kerr instanton and the Taub-bolt instanton, as described above. However, we should also note that the local metrics of the self-dual Taub-NUT and Eguchi–Hanson instantons can also be recovered from (5.41) up to coordinate transformations, by taking very special limits.

5.3.9 Multi-collinearly-centered Taub-NUT instanton

The general multi-Taub-NUT instanton [101, 108] admits only a one-parameter isometry group, but a particular class admits a larger $U(1) \times U(1)$ isometry group,

[8] Moreover, the presence of these orbifold singularities would imply that the bolt at $r = r_0$ itself will not be a completely regular S^2 surface. This surface will in general possess conical singularities at the two poles.

namely those with all the nuts collinearly centered on the base space E^3. The metric of this special class takes the form

$$\mathrm{d}s^2 = H^{-1}\,(\mathrm{d}\psi + A)^2 + H(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{5.45}$$

where the harmonic function H and the twist potential A on E^3 are defined as

$$\begin{aligned} H &= 1 + \sum_{i=1}^{k} \frac{2|n_i|}{r_i}\,, \\ A &= \sum_{i=1}^{k} \frac{2n_i(r\cos\theta - a_i)}{r_i}\,\mathrm{d}\phi\,. \end{aligned} \tag{5.46}$$

Here $r_i = \sqrt{r^2 + {a_i}^2 - 2a_i r\cos\theta}$ is the distance between the i-th nut and the position under consideration on the flat base space E^3 parameterized by the spherical polar coordinates (r, θ, ϕ). Without loss of generality, we assume $0 < a_1 < \cdots < a_k$. There are in total k nuts in the space, collinearly centered on the base space E^3 along the symmetry axis at $(r = a_i, \theta = 0)$ respectively. The parameters n_i and coordinates r, θ take the ranges $-\infty < n_i < \infty$, $r \geq 0$, $0 \leq \theta \leq \pi$. Furthermore, all the NUT charges n_i are required to have the same sign.

The Weyl–Papapetrou coordinates (ψ, ϕ, ρ, z) are related to the above coordinates by

$$\rho = r\sin\theta\,, \qquad z = r\cos\theta\,. \tag{5.47}$$

In these coordinates, the rod structure has k turning points in total, corresponding to the k nuts, located at $(\rho = 0, z = z_i \equiv a_i)$ or $(r = a_i, \theta = 0)$. The direction of the i-th rod is $\ell_i = (-2\sum_{j=1}^{i-1} n_j + 2\sum_{j=i}^{k} n_j, 1)$, for $i = 1, 2, \ldots, k+1$.

It can be checked that the condition for the adjacent rod direction pairs $\{\ell_{i-1}, \ell_i\}$ and $\{\ell_i, \ell_{i+1}\}$, for $i = 2, \ldots, k$, to be related by a $GL(2, \mathbb{Z})$ transformation requires that $n_{i-1} = n_i \equiv n$. Hence, regularity of the gravitational instanton requires all the NUT charges to be equal [102]; this is what we assume from now on. Furthermore, the orbits generated by say the i-th adjacent direction pair $\{\ell_i, \ell_{i+1}\}$ should be identified with period 2π independently. This direction pair is then identified as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group of the multi-collinearly-centered Taub-NUT instanton. If k is odd, we have $\ell_{\lfloor \frac{k}{2} \rfloor + 1} = (2n, 1)$ and $\ell_{\lfloor \frac{k}{2} \rfloor + 2} = (-2n, 1)$, and the identifications (5.7) should be made to ensure regularity; if k is even, we have $\ell_{\frac{k}{2}} = (4n, 1)$ and $\ell_{\frac{k}{2}+1} = (0, 1)$, and the identifications (5.32) should be made to ensure regularity.

If $n \neq 0$, we can put the rod structure in standard orientation by taking $\{\tilde{V}_{(1)} = \ell_{k+1}, \tilde{V}_{(2)} = \ell_1\}$. The corresponding new Weyl–Papapetrou coordinates $(\tilde{\psi}, \tilde{\phi}, \tilde{\rho}, \tilde{z})$ are related to the old coordinates (5.47) by

$$\psi = -2nk(\tilde{\psi} - \tilde{\phi}) \,, \qquad \phi = \tilde{\psi} + \tilde{\phi} \,, \qquad \rho = \frac{1}{4k|n|}\tilde{\rho} \,, \qquad z = \frac{1}{4k|n|}\tilde{z} \,. \tag{5.48}$$

The i-th turning point is now pushed to $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_i \equiv 4|n|ka_i)$, and the directions of the $k+1$ rods from left to right are $K_1 = (0, 1)$, …, $K_i = (\frac{i-1}{k}, \frac{k-i+1}{k})$, …, $K_{k+1} = (1, 0)$. (Fig. 5.3 illustrates the case of $k = 3$.) In the new Weyl–Papapetrou coordinates, the following identifications should be made to ensure regularity:

$$(\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi}, \tilde{\phi} + 2\pi) \,, \qquad (\tilde{\psi}, \tilde{\phi}) \to \left(\tilde{\psi} + \frac{2\pi}{k}, \tilde{\phi} + \frac{2(k-1)\pi}{k}\right) . \tag{5.49}$$

We note that the rod structure of the multi-collinearly-centered Taub-NUT instanton also admits a one-parameter degeneracy as that in the double-centered case discussed in subsection 5.3.6.

Constant r surfaces for $a_i < r < a_{i+1}$ $(i = 1, \ldots, k-1)$ are represented in the (ρ, z)

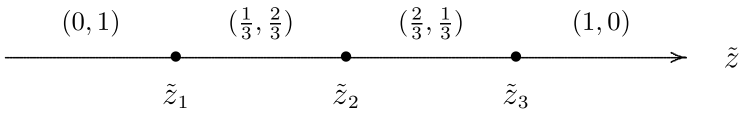

Figure 5.3: The rod structure of the triple-collinearly-centered Taub-NUT instanton in standard orientation.

half-plane of the orbit space by curves intersecting with the first and $(i+1)$-th rods, and since $K_{i+1} = -(i-1)K_1 + iK_2$, these surfaces have lens-space topology $L(i, 1-i) \cong L(i, 1)$. By a similar argument, surfaces of constant $r > a_k$ have topology $L(k, 1-k) \cong L(k, 1)$. At infinity $r \to \infty$, the Killing vector field $\frac{\partial}{\partial \psi}$ generates a compact dimension of constant size $8n\pi$. It can be checked that the multi-collinearly-centered Taub-NUT instanton is ALF with $\Gamma = \mathbb{Z}_k$. When $n \to 0$, we recover a three-dimensional flat space; on the other hand, when $n \to \infty$, the compact dimension blows up and we recover the collinearly-centered Gibbons–Hawking instanton [108].[9] The latter instanton is ALE with $\Gamma = \mathbb{Z}_k$.

The multi-collinearly-centered Taub-NUT instanton considered here, with all the axes at $\rho = 0$ removed, is naturally taken as a $U(1) \times U(1)$-principal fibre bundle over the base space $\hat{I} = \{z + i\rho \in \mathbb{C} \mid \rho > 0\}$. However, in the more conventional but equivalent way, with all the nuts removed, this gravitational instanton is taken as a $U(1)$-principal fibre bundle over the base space E^3 with the corresponding nut-points removed. See, e.g., [118] for this more conventional point of view.

[9] The usual form of the Gibbons–Hawking instanton is to omit the 1 in the harmonic function H [108], but this is equivalent to the limit described above by taking $n \to \infty$.

5.4 Possible new gravitational instantons

5.4.1 Possible new gravitational instantons with two turning points

We have analyzed the rod structure of five classes of regular gravitational instantons with two turning points, namely the Euclidean Schwarzschild, Euclidean Kerr, Eguchi–Hanson, double-centered Taub-NUT and Taub-bolt instantons. It may be interesting to ask what kind of rod structure with two turning points is necessary for a regular solution.

As we have seen, any direction pair of adjacent rods, say $\{\ell_1, \ell_2\}$, should generate orbits with period 2π independently to avoid possible conical and orbifold singularities, and they are then identified as the pair of independent 2π-periodic generators of the $U(1)\times U(1)$ isometry group of the space. Any other rod direction should be a linear combination of them with coprime integer coefficients, i.e., $\ell_3 = q\ell_1 + p\ell_2$ with coprime integers p and q.[10] Furthermore, the pair $\{\ell_2, \ell_3\}$ should also be related to the pair $\{\ell_1, \ell_2\}$ by a $GL(2, \mathbb{Z})$ transformation, so we have $q = \pm 1$. Depending on the value of p, we further divide the possibilities into two classes: (a) $p = 0$, so $\ell_1 = \pm\ell_3$, or (b) $|p| \geq 1$, so $\ell_2 = (\ell_1 \pm \ell_3)/p$. The first class corresponds to the case when the two semi-infinite rods are parallel, while the second class corresponds to the case when they are not.

[10] The case when $\ell_1 = \ell_2$ or $\ell_2 = \ell_3$ is trivial in our analysis, as it results in a rod structure with one turning point. We have already mentioned in subsection 5.3.8 that the self-dual Taub-NUT metric can be recovered from the Kerr-bolt metric by imposing this condition.

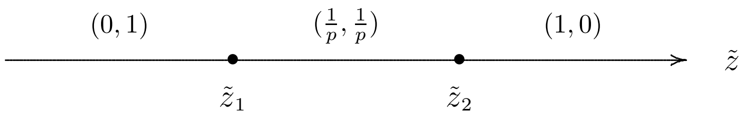

Figure 5.4: The rod structure of possible new gravitational instantons with two turning points, for integer $|p| \geq 3$, in standard orientation.

The Euclidean Schwarzschild and Kerr instantons are in class (a). The Taub-bolt instanton is in class (b) with $|p| = 1$, while the Eguchi–Hanson and double-centered Taub-NUT instantons are in class (b) with $|p| = 2$. If there exists a solution in class (b) with an integer $|p| \geq 3$ (the rod structure for such a solution in standard orientation is illustrated in Fig. 5.4), it would be possible to remove the conical and orbifold singularities, by identifying the orbits generated by say $\{\ell_1, \ell_2\}$ with period 2π independently. Asymptotically it will have a lens-space $L(p, 1)$ structure, with or without any compact dimensions. This class of possible new gravitational instantons was previously considered in [107]. In that paper, it was also argued that topological constraints, in the form of a Hitchin-type inequality, might rule out such new gravitational instantons for sufficiently large values of $|p|$. However, there are still some small values of $|p| \geq 3$ for which new ALF or ALE gravitational instantons are not ruled out.

5.4.2 Possible new gravitational instantons with three turning points

It is straightforward to extend this analysis to find all possible rod structures with any given number of turning points, that would correspond to regular manifolds if appropriate identifications are made. In this subsection, we illustrate this for the next simplest case of three turning points. Again, we find that the allowed rod structures fall into two classes, depending on whether the two semi-infinite rods are parallel or not. The first class, with rod structure in standard orientation as shown in Fig. 5.5(a), has parallel semi-infinite rods. The infinity of the associated gravitational instanton will have topology $S^1 \times S^2$, so it will likely be AF. The second class has rod structure in standard orientation as shown in Fig. 5.5(b), for any pair of integers p and q except $p = q = \pm 1$. Asymptotically it will have a lens-space $L(pq-1, q)$ structure, with or without any compact dimensions.

Note that the triple-collinearly-centered Taub-NUT instanton (Fig. 5.3) emerges as a special case of the second class with $p = q = \pm 2$. However, to the best of our knowledge, all the other cases would be associated to new gravitational instantons, if they exist. A systematic search is currently underway to construct new gravitational instantons with one or more of these allowed rod structures; indeed, preliminary results indicate that a new AF gravitational instanton with rod structure belonging to the first class (Fig. 5.5(a)) exists [119]. For the second class, topological or other constraints might serve to rule out certain ranges of values of p and q. However, there might still exist new ALF or ALE gravitational instantons with sufficiently small values of $|p|$ and $|q|$.

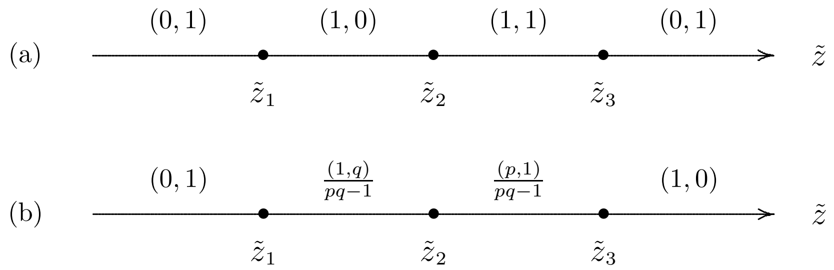

Figure 5.5: The rod structure of possible new gravitational instantons with three turning points in standard orientation. In (b), p and q are any pair of integers except $p = q = \pm 1$. Note that the special case $p = q = \pm 2$ corresponds to the triple-collinearly-centered Taub-NUT instanton (Fig. 5.3).

5.5 Discussion

In this chapter, we applied the rod-structure formalism to study gravitational instantons with $U(1) \times U(1)$ isometry. A number of examples were considered, and several previous results concerning certain of these gravitational instantons were clarified in the process. It was then argued that the rod structure provides a way to classify all possible gravitational instantons with $U(1) \times U(1)$ isometry, and new classes of rod structures were explicitly written down. We finally speculated that at least some of these new rod structures would be associated to as yet undiscovered gravitational instantons.

It is interesting to further explore the (non-)uniqueness of the gravitational instantons analyzed in this chapter. When restricted to the case when there is no black hole, Hollands and Yazadjiev's theorems [38, 39] immediately imply the following result: For gravitational instantons with $U(1) \times U(1)$ isometry, asymptotically approaching the Euclidean space E^4, or the product space $E^3 \times S^1$ (with S^1 finite), there exists at most one gravitational instanton for a given rod structure. At infinity, the $U(1) \times U(1)$ isometry is assumed to generate the standard rotations of E^4, or of E^3 and S^1. It immediately follows that four-dimensional flat space and the Euclidean Schwarzschild instanton are the unique gravitational instantons that asymptotically approach E^4 and $E^3 \times S^1$ respectively for their corresponding rod structures.

We have seen in section 5.3 that, in certain cases, a gravitational instanton with $U(1) \times U(1)$ isometry cannot be uniquely determined by its rod structure. On the contrary, we even find that a one-parameter family of gravitational instantons can share the same rod structure. The simplest example is given by four-dimensional flat space and the self-dual Taub-NUT instanton, which share the same rod structure in standard orientation (as shown in Fig. 5.1). However, we notice that in this example the one-parameter degeneracy is resolved once the NUT charge of the self-dual Taub-NUT instanton, and thus its asymptotic geometry, is specified. Together with the result in the preceding paragraph, we may naturally conjecture that a gravitational instanton can be uniquely determined by its rod structure together with its asymptotic geometry, if specified in some appropriate way. Here, we also assume the various technical assumptions made in [38, 39] but with a vanishing black hole horizon holding in our case. One may further expect that this conjecture, if true, can be proved similarly as was done for the theorems in [38, 39].

It would be interesting to know why the rod structure alone does not contain sufficient information to determine a gravitational instanton. Insights may be gained by studying the sources [37], and the relations between these sources and the corresponding rod structures of the gravitational instantons analyzed in this chapter.

The open problem regarding the existence of a gravitational instanton for a given rod structure is even more challenging. Firstly, up to now not all the rod structures allowed by our analysis (so that conical and orbifold singularities can be removed) have been associated with a gravitational instanton, neither has their existence been disproved in general. For some of the new rod structures that we considered in section 5.4, topological or other constraints might rule out the existence of any new associated gravitational instantons. If, however, these new gravitational instantons do exist, the inverse scattering method [46, 47] is a powerful solution-generating technique that might be able to construct them. Secondly, for a given rod structure, supposing their associated gravitational instantons exist, there seems to be constraints on the asymptotic geometry of these gravitational instantons. For example, gravitational instantons with a rod structure as shown in Fig. 5.2(c) (the Taub-bolt instanton) are only found to be ALF. An ALE gravitational instanton with such a rod structure cannot exist. This is because such a new gravitational instanton will have to be trivially AE (it has an infinity of topology S^3), and the positive action theorem [114] rules out this possibility. On the other hand, gravitational instantons with a rod structure as shown in Fig. 5.2(b) can be either ALF (the double-centered Taub-NUT instanton) or ALE (the Eguchi–Hanson instanton).

Notice that a flat time dimension can be trivially added to the above gravitational instantons to obtain five-dimensional space-time solutions to the vacuum Einstein equations. Moreover, static or stationary black holes may be added to such space-times, while preserving the $U(1) \times U(1)$ isometry. The black holes are then said to be sitting on that corresponding gravitational instanton, since when they are removed we recover a direct product of that gravitational instanton and a flat time dimension. Indeed, we have been able to classify and construct black holes on almost all the gravitational instantons studied in this chapter with one or two turning points. This is the subject of the next chapter.

The rod-structure formalism developed in this thesis may be readily generalized, in some modified form, to gravitational instantons with cosmological constant $\Lambda > 0$. These instantons are compact Einstein manifolds [102]. All the explicitly known gravitational instantons within this class, namely S^4, $\mathbb{C}P^2$, $S^2 \times S^2$ and $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ with their corresponding Einstein metrics [102, 111, 120], have the prescribed $U(1) \times U(1)$ isometry group. The concepts of turning points and rods are still applicable as fixed points of the $U(1) \times U(1)$ isometry group and its $U(1)$ isometry subgroups respectively. The direction of a rod is the normalized 2π-periodic generator of the $U(1)$ isometry subgroup which (as a Killing vector field) vanishes along that rod. Two adjacent rods intersect at a turning point, with directions satisfying the constraint (3.10). The direction pair of any two adjacent rods can then be identified as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group. It turns out that the orbit spaces of these compact gravitational instantons are homeomorphic to a disk [84, 85]. The boundary, which is homeomorphic to a circle, is divided into arcs by the turning points. An arc on this circle is what was roughly referred to as a rod above. S^4, as an analytic continuation of the de Sitter space-time [121], and $\mathbb{C}P^2$ [122, 123] have two and three turning points, respectively. Both $S^2 \times S^2$ [122] and $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ [124] have four turning points. The rod directions of these gravitational instantons can be easily calculated, and it turns out that the rod structures of $S^2 \times S^2$ and $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$ can distinguish between these two gravitational instantons. We do not give any more details here.

Chapter 6

Black holes on gravitational instantons

In this chapter, we classify and construct black holes, with $\mathbb{R} \times U(1)^2$ isometry, on gravitational instantons. These black holes are five-dimensional, completely regular space-times, whose spatial backgrounds are gravitational instantons with $U(1)\times U(1)$ isometry. Most of the known exact five-dimensional black hole solutions can be classified within this scheme. Amongst the new solutions that are presented are static black holes on the Euclidean Kerr and Taub-bolt instantons. We also present a rotating black hole on the Eguchi–Hanson instanton.

6.1 Introduction

The majority of the known black hole solutions to date, in both four and higher dimensions, are asymptotically flat in the sense that they approach a Minkowski space-time $M^{1,n}$ at infinity. This is physically expected of an isolated gravitating system. And indeed, when the black hole is removed from the space-time (for example, by setting its mass to zero), the resulting background space-time is nothing but a direct product of E^n and a flat time dimension.

In higher dimensions, however, black holes can admit a variety of other asymptotic behaviour, while still becoming "flat" at infinity. In five dimensions, for example, it is possible to consider black hole solutions that are asymptotically $M^{1,3} \times S^1$, or a finite but non-trivial S^1 fiber bundle over $M^{1,3}$ at infinity. Such solutions are of interest in Kaluza–Klein theory where the fifth dimension is assumed to be compactified into a circle. Examples of such solutions are black holes/rings on Taub-NUT space. When the black hole/ring is removed, the resulting background space-time is a direct product of Taub-NUT space and a flat time dimension.

In general, the background of a five-dimensional black hole space-time can be the direct product of any regular Ricci-flat four-manifold, which is, in fact, a gravitational instanton, and a flat time dimension. Conversely, for any given gravitational instanton, we can add a trivial flat time dimension to it to obtain five-dimensional space-times as solutions to the vacuum Einstein equations.[1] Moreover, stationary

[1] Such space-times, when constructed with AF/ALF gravitational instantons, have been studied in the context of Kaluza–Klein theory; see, e.g., [125, 126] for details. When dimensionally reduced to four dimensions, they describe magnetic monopoles or dipoles, or their superpositions.

(static as a special case) black holes may be added into such space-times. The black holes are then said to be sitting on that corresponding instanton, as when they are removed, we recover a direct product of a flat time dimension and that instanton. The asymptotic behaviour of the gravitational instanton is directly related to the asymptotic behaviour of a black hole on this gravitational instanton. Black holes on AE gravitational instantons will be asymptotically flat in the usual sense, whereas black holes on ALE gravitational instantons will be asymptotically locally flat, i.e., they approach $M^{1,4}/\mathbb{Z}_p$ for some integer p at infinity. Black holes on AF or ALF gravitational instantons will be asymptotically Kaluza–Klein. A common feature of asymptotically Kaluza–Klein space-times is that there exists at least one compact dimension at infinity.

So far, there has only been limited success in constructing black holes on gravitational instantons other than four-dimensional flat space and Taub-NUT space. One of the few known ways to systematically construct black holes on a non-trivial gravitational instanton is within some five-dimensional supergravity theory, say $N = 1$ minimal supergravity. In such a theory, an underlying linear structure allows the superposition of supersymmetric black holes on any four-dimensional hyper-Kähler manifold [68], which includes the Taub-NUT space and the Eguchi–Hanson instanton. This was the method used to construct black holes/rings on the latter gravitational instanton in [127, 128]. In a similar fashion, extremal black holes on the Eguchi–Hanson instanton [129] and the multi-Taub-NUT instanton [94] have been constructed in five-dimensional Einstein–Maxwell theory.

In this chapter, we are interested in black holes on gravitational instantons in vacuum Einstein gravity. The construction of such solutions presents more difficulty than in the supersymmetric or Einstein–Maxwell case, although progress is still possible if a $\mathbb{R} \times U(1)^2$ isometry is assumed, which will be adopted from now on in this chapter. For these solutions, the background gravitational instantons will possess an isometry group $U(1) \times U(1)$, and they have been classified in terms of the rod structure in the chapter 5. In this chapter, we will classify and construct black holes on these gravitational instantons. Most of the exact five-dimensional vacuum black holes with zero cosmological constant in the current literature can be classified within this scheme.

In the following sections, we will classify/construct black holes on four-dimensional flat space, the self-dual Taub-NUT, Euclidean Schwarzschild, Euclidean Kerr, Eguchi–Hanson and Taub-bolt instantons respectively. The static black holes on the Euclidean Kerr and Taub-bolt, and rotating black hole on the Eguchi–Hanson instantons constructed here are new completely regular five-dimensional spacetimes. As already mentioned, the solutions of this chapter will have an isometry group $\mathbb{R} \times U(1)^2$, so we will analyze their rod structures. All the metrics of these solutions are independent of three coordinates (t, ψ, ϕ). Hence we naturally take $\{V_{(0)} = \frac{\partial}{\partial t}, V_{(1)} = \frac{\partial}{\partial \psi}, V_{(2)} = \frac{\partial}{\partial \phi}\}$, and define the corresponding Weyl–Papapetrou coordinates $(x^0 = t, x^1 = \psi, x^2 = \phi, \rho, z)$. The direction of a rod is written in the form (a_0, a_1, a_2) for simplicity, which, in fact, is $a_0 \frac{\partial}{\partial t} + a_1 \frac{\partial}{\partial \psi} + a_2 \frac{\partial}{\partial \phi}$. New Weyl–Papapetrou coordinates $(\tilde{x}^0 = \tilde{t}, \tilde{x}^1 = \tilde{\psi}, \tilde{x}^2 = \tilde{\phi}, \tilde{\rho}, \tilde{z})$ are also introduced, in which the rod structure of the black hole in standard orientation is defined.

6.2 Black holes on four-dimensional flat space

The five-dimensional Minkowski space-time is obtained by adding a flat time dimension to four-dimensional flat space. The simplest black hole sitting on the four-dimensional flat space is then the five-dimensional Schwarzschild black hole [29]. In spherical polar coordinates it has the following form:

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r^2}\right)\mathrm{d}t^2 + \left(1-\frac{2m}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\psi^2 + \cos^2\theta\,\mathrm{d}\phi^2\right), \tag{6.1}$$

where the parameter m and coordinates t, r, θ take the ranges $m \geq 0$, $-\infty < t < \infty$, $r \geq r_0 \equiv \sqrt{2m}$, $0 \leq \theta \leq \frac{\pi}{2}$. The black hole horizon and physical infinity are located at $r = r_0$ and $r = \infty$, respectively.

The Weyl–Papapetrou coordinates (t, ψ, ϕ, ρ, z) are related to the above coordinates by

$$\rho = \frac{1}{2}r^2\sqrt{1-\frac{2m}{r^2}}\sin 2\theta\,, \qquad z = \frac{1}{2}r^2\left(1-\frac{m}{r^2}\right)\cos 2\theta\,. \tag{6.2}$$

In these coordinates, the rod structure has two turning points, at $(\rho = 0, z = z_1 \equiv -\frac{m}{2})$ or $(r = r_0, \theta = \frac{\pi}{2})$, and $(\rho = 0, z = z_2 \equiv \frac{m}{2})$ or $(r = r_0, \theta = 0)$. These two points correspond to the south and north poles of the black hole horizon. From left to right, the three rods are:

- Rod 1: a semi-infinite space-like rod located at $(\rho = 0, z \leq z_1)$ or $(r \geq r_0, \theta = \frac{\pi}{2})$, with direction $\ell_1 = (0, 0, 1)$.
- Rod 2: a finite time-like rod located at $(\rho = 0, z_1 \leq z \leq z_2)$ or $(r = r_0, 0 \leq \theta \leq \frac{\pi}{2})$, with direction $\ell_2 = \frac{1}{\kappa}(1, 0, 0)$, where $\kappa = \frac{1}{\sqrt{2m}}$.

- Rod 3: a semi-infinite space-like rod located at ($\rho = 0, z \geq z_2$) or ($r \geq r_0, \theta = 0$), with direction $\ell_3 = (0, 1, 0)$.

This rod structure is illustrated in Fig. 6.1. The second rod is time-like and represents the black hole horizon. The two semi-infinite rods 1 and 3 are space-like, and represent the two asymptotic axes.

We can identify $\{\ell_1, \ell_3\}$ with period 2π independently to ensure regularity of the space-time. However, it is interesting to note that, since there are no two space-like rods intersecting at a turning point, we can further identify a third Killing vector field $\frac{1}{s}(q\ell_1 + p\ell_3)$ with period 2π without introducing neither conical nor orbifold singularities, provided that s, p and q are mutually coprime non-zero integers. The resulting black hole space-time has the same topology of asymptotic structure as that of the event horizon, i.e., a lens-space $L(s; q, p)$ [109]. By removing the black hole from this space-time (setting $m = 0$), we recover a quotiented five-dimensional Minkowski space-time background $M^{1,4}/\mathbb{Z}_{|s|}$, which is singular if $|s| \geq 2$, as there will be a $\mathbb{Z}_{|s|}$ orbifold singularity present at the origin.

In this chapter, $\{\ell_1, \ell_3\}$ are identified with period 2π independently, so the following identifications

$$(\psi, \phi) \to (\psi, \phi + 2\pi)\,, \qquad (\psi, \phi) \to (\psi + 2\pi, \phi)\,, \tag{6.3}$$

are made to ensure the regularity of the black hole space-time (6.1), as well as its background. The resulting space-time is called the five-dimensional Schwarzschild black hole, whose $U(1) \times U(1)$ isometry group is generated by the two independent 2π-periodic generators $\{\ell_1, \ell_3\}$. It is then obvious that the five-dimensional

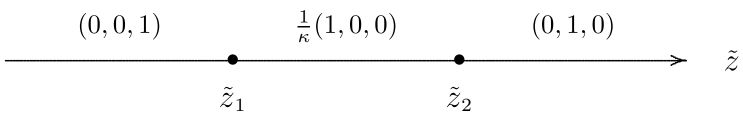

Figure 6.1: The rod structure of the five-dimensional Schwarzschild black hole and the Ishihara–Matsuno black hole.

Schwarzschild black hole has a horizon topology S^3, and is asymptotically flat.

Angular momenta can be added to the black hole along the two orthogonal axes at infinity, giving the five-dimensional Myers–Perry black hole [18]. In this case, the direction of the time-like rod is rotated, i.e., it has components involving ℓ_1 and ℓ_3, so the black hole has non-vanishing angular velocities. The rod structure of the five-dimensional Myers–Perry black hole has been thoroughly studied in subsection 3.1.3.

The five-dimensional Schwarzschild and Myers–Perry black holes sit at the turning point of the four-dimensional flat space (thus covering it). A black hole can also sit, say, on the left semi-infinite rod, of the four-dimensional flat space. In terms of the rod structure, this can be done by cutting the left semi-infinite rod, and placing a time-like rod there. Angular momenta could also be added to the black hole to balance the gravitation. The resulting solutions are the Emparan–Reall [19] and Pomeransky–Sen'kov [49] black rings. We can perform a series of these operations, and thus get the black saturn [50], black di-ring [51, 52] and black bi-ring [53, 54] solutions. By removing the black holes/rings in these configurations, we recover the five-dimensional Minkowski space-time.

6.3 Black holes on the self-dual Taub-NUT instanton

The well-known Gross–Perry–Sorkin magnetic monopole [125, 126], also known as the Kaluza–Klein monopole, when lifted to five dimensions in a regular fashion, is nothing but the five-dimensional space-time obtained by adding a flat time dimension to the self-dual Taub-NUT instanton. The simplest black hole sitting on the self-dual Taub-NUT instanton is then the magnetically charged static Kaluza–Klein black hole, lifted to five dimensions. In the form given by Ishihara and Matsuno [60], it takes the following metric:

$$\mathrm{d}s^2 = \frac{r^2}{4}\left(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\right)^2 - \left(1-\frac{2m}{r^2}\right)\mathrm{d}t^2 + \frac{k(r)^2}{1-\frac{2m}{r^2}}\mathrm{d}r^2 + \frac{k(r)r^2}{4}\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{6.4}$$

where $k(r)$ is a function defined as

$$k(r) = \frac{b^2(b^2-2m)}{(b^2-r^2)^2}\,. \tag{6.5}$$

The parameters m, b and coordinates t, r, θ take the ranges $-\infty < t < \infty$, $\sqrt{2m} \leq r \leq b$, and $0 \leq \theta \leq \pi$. The black hole horizon and physical infinity are located at $r = r_0 \equiv \sqrt{2m}$ and $r = b$, respectively.

The Weyl–Papapetrou coordinates (t, ψ, ϕ, ρ, z) are related to the above coordinates by[2]

$$\rho = \frac{br\sqrt{(r^2-2m)(b^2-2m)}}{4(b^2-r^2)}\sin\theta\,, \qquad z = \frac{b^2r^2 - mr^2 - mb^2}{4(b^2-r^2)}\cos\theta\,. \tag{6.6}$$

In these coordinates, the rod structure has two turning points, at $(\rho = 0, z = z_1 \equiv -\frac{m}{4})$ or $(r = r_0, \theta = \pi)$, and $(\rho = 0, z = z_2 \equiv \frac{m}{4})$ or $(r = r_0, \theta = 0)$. These two

[2]Note that in these Weyl–Papapetrou coordinates $V_{(0)} = \frac{\partial}{\partial t}$ is not normalized at infinity.

points correspond to the south and north poles of the black hole horizon. From left to right, the three rods are:

- Rod 1: a semi-infinite space-like rod located at $(\rho = 0, z \leq z_1)$ or $(r \geq r_0, \theta = \pi)$, with direction $\ell_1 = (0, 1, 1)$.
- Rod 2: a finite time-like rod located at $(\rho = 0, z_1 \leq z \leq z_2)$ or $(r = r_0, 0 \leq \theta \leq \pi)$, with direction $\ell_2 = \frac{1}{\kappa}(\frac{b}{\sqrt{b^2-2m}}, 0, 0)$, where $(\frac{b}{\sqrt{b^2-2m}}, 0, 0)$ is the Killing vector field corresponding to time flow normalized at infinity, and $\kappa = \sqrt{\frac{1}{2m} - \frac{1}{b^2}}$ is the surface gravity on the horizon for this Killing vector field.
- Rod 3: a semi-infinite space-like rod located at $(\rho = 0, z \geq z_2)$ or $(r \geq r_0, \theta = 0)$, with direction $\ell_3 = (0, -1, 1)$.

Rod 2 is time-like and represents the black hole horizon. To ensure regularity, we identify $\{\ell_1, \ell_3\}$ with period 2π independently, which can then be taken as the two independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group of the space-time, though in general, as in the case of the five-dimensional Schwarzschild black hole studied in the previous section, the space-time could further be quotiented by a $\mathbb{Z}_s$ group for any natural number s. Thus the following identifications on the coordinates (ψ, ϕ) are made to ensure regularity:

$$(\psi, \phi) \to (\psi + 4\pi, \phi)\,, \qquad (\psi, \phi) \to (\psi + 2\pi, \phi + 2\pi)\,. \tag{6.7}$$

We can put the rod structure in standard orientation by taking $\{\tilde{V}_{(0)} = \frac{b}{\sqrt{b^2-2m}}\frac{\partial}{\partial t}, \tilde{V}_{(1)} = \ell_2, \tilde{V}_{(2)} = \ell_1\}$. The corresponding new Weyl–Papapetrou coordinates $(\tilde{t}, \tilde{\psi}, \tilde{\phi}, \tilde{\rho}, \tilde{z})$

are related to the old coordinates (6.6) by:

$$t = \frac{b}{\sqrt{b^2-2m}}\,\tilde{t}\,, \quad \psi = -\tilde{\psi}+\tilde{\phi}\,, \quad \phi = \tilde{\psi}+\tilde{\phi}\,, \quad \rho = \frac{\sqrt{b^2-2m}}{2b}\,\tilde{\rho}\,, \quad z = \frac{\sqrt{b^2-2m}}{2b}\,\tilde{z}\,. \tag{6.8}$$

The two turning points are now pushed to $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_1 \equiv -\frac{mb}{2\sqrt{b^2-2m}})$ and $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_2 \equiv \frac{mb}{2\sqrt{b^2-2m}})$, and the corresponding directions of the three rods from left to right are $K_1 = (0,0,1)$, $K_2 = \frac{1}{\kappa}(1,0,0)$ and $K_3 = (0,1,0)$. This is illustrated in Fig. 6.1. It should be noted that the new time coordinate $\tilde{t}$ is normalized at infinity. In the new Weyl–Papapetrou coordinates, the following identifications are made to ensure regularity:

$$(\tilde{\psi},\tilde{\phi}) \to (\tilde{\psi},\tilde{\phi}+2\pi)\,, \qquad (\tilde{\psi},\tilde{\phi}) \to (\tilde{\psi}+2\pi,\tilde{\phi})\,. \tag{6.9}$$

We can see that this solution is fully characterized by its rod structure, i.e., given any positive value of $\tilde{z}_2$ and κ satisfying $\kappa^2\tilde{z}_2 \leq \frac{1}{4}$, we can find only one solution corresponding to these parameters. The degeneracy of rod structure only occurs when the black hole is absent, i.e., in the background space-time. The background space-time is obtained by taking $r_0 = 0$, which is nothing but the direct product of a flat time dimension and the self-dual Taub-NUT instanton in a new form.

It is obvious that the black hole has a horizon topology S^3. At infinity $r \to b$, the space-time approaches the direct product of a flat time dimension and the asymptotic structure of the self-dual Taub-NUT instanton with NUT charge $n = \frac{b}{4}$ [60]. $\frac{\partial}{\partial\psi}$ generates the compact dimension at infinity, with a constant size $2\pi b$. By taking the NUT charge to infinity, thus $b \to \infty$ and $k(r) = 1$, we obviously recover the five-dimensional Schwarzschild black hole.

Angular momenta can be added to the solution (6.4) [61], which, when dimensionally reduced to four dimensions, has the interpretation of a static dyonic Kaluza–Klein black hole [130–133]. In our context, its rod structure can be easily analyzed, and it can be classified as a stationary black hole on the self-dual Taub-NUT instanton with one rotational parameter. The general stationary black hole on the self-dual Taub-NUT instanton with two independent rotational parameters is then the rotating dyonic Kaluza–Klein black hole [91, 134, 135], if lifted to five dimensions with the regularity conditions imposed appropriately (see chapter 7 for more details). By taking the NUT charge of the space-time to infinity, the five-dimensional Myers–Perry black hole is recovered [136–138]. In these cases, the direction of the time-like rod is rotated, i.e., it has components involving ℓ_1 and ℓ_3, so the black hole has non-vanishing angular velocities.

The solution of the general rotating dyonic Kaluza–Klein black hole lifted to five dimensions appropriately, as a regular five-dimensional space-time, describes a stationary black hole sitting at the turning point of the self-dual Taub-NUT instanton [136–138]. In terms of the rod structure, a black hole can also sit on one of the semi-infinite rods of the self-dual Taub-NUT instanton, resulting in a black ring solution. The static class was constructed by Ford et al. [62], which is singular. By adding angular momenta to balance the gravitation, Camps et al. constructed a regular rotating black ring on the self-dual Taub-NUT instanton [63]. But note that their solution is not the most general solution in this class, as it has only one rotational parameter. A more general solution with two independent rotational parameters, classified in our scheme as a double-rotating black ring on the self-dual Taub-NUT instanton, is expected to exist, but has not been constructed explicitly.

If such a solution exists, we would be able to recover the Pomeransky–Sen'kov black ring [49] by taking the NUT parameter to infinity. Also multi-black hole solutions on the self-dual Taub-NUT instanton may exist. In all these configurations, by removing the black holes/rings, we recover the self-dual Taub-NUT instanton with a flat time dimension.

We emphasize that the NUT charge of the above solutions characterizes the size of the compact (space-like) dimension at infinity for these space-times, which are completely regular as five-dimensional solutions. It is quite clear that, for all the solutions classified in this subsection, asymptotically there are no cross terms of the time and any of the spatial directions. Thus if we perform the Kaluza–Klein reduction along the compact dimension, we obtain four-dimensional asymptotically flat space-times, without NUT charges. In this sense, the NUT charge considered here is really a characterization of the four-dimensional spatial sections, rather than the five-dimensional space-time itself. So the time dimension is non-compact, and has the range $-\infty < t < \infty$. There indeed exist solutions which have cross terms of the time and other spatial directions. If the Kaluza–Klein reduction is performed on these solutions, we obtain four-dimensional space-times with real NUT charges, so in these cases, the time dimension is probably identified and may be compact. In fact, the Kaluza–Klein black holes in [91, 133, 135] were obtained by imposing the condition that these real NUT charges vanish.

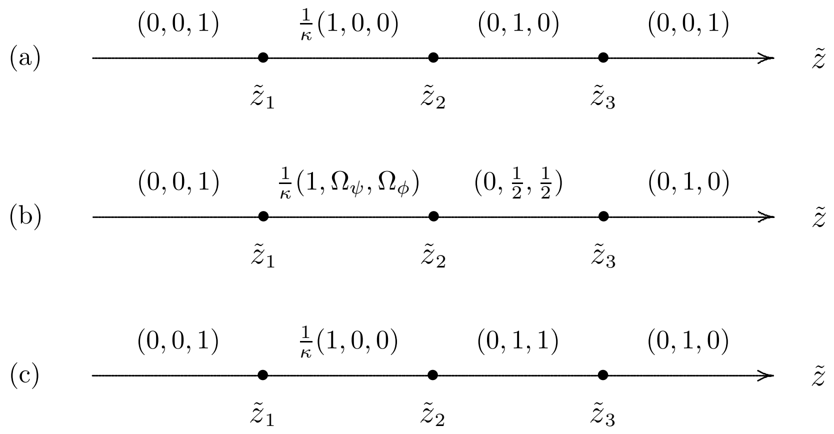

Figure 6.2: The rod structure of: (a) a black hole on the Euclidean Schwarzschild or Kerr instanton; (b) a (rotating) black hole on the Eguchi–Hanson instanton; and (c) a black hole on the Taub-bolt instanton; all in standard orientation.

6.4 Black holes on the Euclidean Schwarzschild instanton

The space-time obtained by adding a flat time dimension to the Euclidean Schwarzschild instanton describes the static Kaluza–Klein bubble of nothing [139]. Black holes can be placed at one or both of the turning points of the Euclidean Schwarzschild instanton. Static solutions of this class were constructed by Emparan and Reall [35] (with a rod structure shown in Fig. 6.2(a)), and Elvang and Horowitz [55]. Rotating solutions were constructed by Tomizawa et al. [57, 58]. In terms of the rod structure, the finite rod of the Euclidean Schwarzschild instanton can also be cut and placed with black holes [56]. In this series of works, these configurations are described as black holes on Kaluza–Klein bubbles. More recently, a rotating black ring on Kaluza–Klein bubbles was constructed [59]. Asymptotically all of these space-times approach $M^{1,3} \times S^1$, with a finite and constant S^1 at infinity. By removing the black holes/rings from these space-times, we recover the Euclidean Schwarzschild instanton with a flat time dimension.

6.5 Black holes on the Euclidean Kerr instanton

The solution describing a static black hole sitting on the Euclidean Kerr instanton has the following metric in C-metric coordinates:[3]

$$\begin{aligned}
\mathrm{d}s^2 &= -\frac{1+cy}{1+cx}\,\mathrm{d}t^2 + \frac{F(x,y)}{H(x,y)}\,(\mathrm{d}\psi+\Omega)^2 \\
&\quad + \frac{2\varkappa^4\,(1+cx)\,H(x,y)}{c^2\,(1-c)\,(1-\alpha^2)\,(1+\alpha^2)^2\,(x-y)^3}\left(\frac{\mathrm{d}x^2}{G(x)} - \frac{\mathrm{d}y^2}{G(y)} + A\,\mathrm{d}\phi^2\right),
\end{aligned} \tag{6.10}$$

where Ω and A are given by

$$\begin{aligned}
\Omega &= \frac{2\alpha c^2\varkappa^2\,[1+c-(1-c)\,\alpha^2][1-c-(1+c)\,\alpha^2]}{(1+\alpha^2)} \\
&\quad \times \frac{(1+y)\,\big[(1-y)\,(2-c+cx)+(1-x)\,(2-c+cy)\,\alpha^2\big]G(x)}{(1-x)\,(x-y)\,F(x,y)}\,\mathrm{d}\phi\,, \\
A &= -\frac{2c^2\,(1-c)\,(1-\alpha^2)\,(1+\alpha^2)^2\,G\,(x)\,G(y)}{(x-y)\,(1+cy)\,F(x,y)}\,.
\end{aligned} \tag{6.11}$$

The functions $G(x)$, $H(x,y)$ and $F(x,y)$ are defined as

$$\begin{aligned}
G(x) &= (1+cx)\,(1-x^2)\,, \\
H(x,y) &= (1+cx)\,\big[\,(1-c)\,\big(1+c-(1-c)\,\alpha^2\big)-(1+cy)\,\big(1-c-(1+c)\,\alpha^2\big)\,\big]^2 \\
&\quad -\alpha^2\,(1+cy)\,\big[\,(1-c)\,\big(1-c-(1+c)\,\alpha^2\big)-(1+cx)\,\big(1+c-(1-c)\,\alpha^2\big)\,\big]^2,
\end{aligned}$$

[3] This solution can be obtained from (5.20) in Ford et al. [62] by imposing $v_< = v_>$. Redefinition of parameters and coordinates are then needed to bring the solution to exactly the same form as above.

$$F(x,y) = \frac{c^2(1-\alpha^2)(1+cx)}{1+c}\left[(1-c)(1-x)(1-y)\left(1-c-(1+c)\alpha^2\right)\right. \\ \left.\times\left(1+c-(1-c)\alpha^2\right)-8\alpha^2(c+x+y+cxy)\right]. \quad (6.12)$$

The parameters $\varkappa$, c, α and coordinates t, x, y take the ranges $\varkappa > 0$, $0 \leq c < 1$, $\alpha^2 < \frac{1-c}{1+c}$, $-\infty < t < \infty$, $-1 \leq x \leq 1$ and $-\frac{1}{c} \leq y \leq -1$.

The Weyl–Papapetrou coordinates (t, ψ, ϕ, ρ, z) are related to the above C-metric coordinates by

$$\rho = \frac{2\varkappa^2\sqrt{-G(x)G(y)}}{(x-y)^2}, \qquad z = \frac{\varkappa^2(1-xy)(2+cx+cy)}{(x-y)^2}. \quad (6.13)$$

In these coordinates, the rod structure has three turning points, at $(\rho = 0, z = z_1 \equiv -c\varkappa^2)$ or $(x = -1, y = -\frac{1}{c})$, $(\rho = 0, z = z_2 \equiv c\varkappa^2)$ or $(x = 1, y = -\frac{1}{c})$, and $(\rho = 0, z = z_3 \equiv \varkappa^2)$ or $(x = 1, y = -1)$. They divide the z-axis into four rods:

- Rod 1: a semi-infinite space-like rod located at $(\rho = 0, z \leq z_1)$ or $(x = -1, -\frac{1}{c} \leq y < -1)$, with direction $\ell_1 = (0,0,1)$.

- Rod 2: a finite time-like rod located at $(\rho = 0, z_1 \leq z \leq z_2)$ or $(-1 \leq x \leq 1, y = -\frac{1}{c})$, with direction $\ell_2 = \frac{1}{\kappa}(1,0,0)$, where

$$\kappa = \frac{(1+\alpha^2)\sqrt{2c(1+c)(1-\alpha^2)}}{4c\varkappa^2(1+c-(1-c)\alpha^2)}, \quad (6.14)$$

  is the surface gravity on the second rod.

- Rod 3: a finite space-like rod located at $(\rho = 0, z_2 \leq z \leq z_3)$ or $(x = 1, -\frac{1}{c} \leq y \leq -1)$, with direction $\ell_3 = (0, \frac{1}{\kappa_E}, \frac{\Omega_E}{\kappa_E})$, where Ω_E and κ_E are defined as

$$\begin{aligned} \Omega_E &= \frac{\alpha(1-\alpha^2)(1+\alpha^2)}{\varkappa^2(1-c-(1+c)\alpha^2)(1+c-(1-c)\alpha^2)}, \\ \kappa_E &= \frac{(1-\alpha^2)(1+\alpha^2)^2\sqrt{1-c^2}}{2\varkappa^2(1-c-(1+c)\alpha^2)(1+c-(1-c)\alpha^2)}. \end{aligned} \quad (6.15)$$

- Rod 4: a semi-infinite space-like rod located at $(\rho = 0, z \geq z_3)$ or $(-1 < x \leq 1, y = -1)$, with direction $\ell_4 = (0, 0, 1)$.

Rod 2 is time-like and represents the black hole horizon. To ensure regularity, we identify the direction pair $\{\ell_3, \ell_4\}$ of adjacent space-like rods with period 2π independently, which can then be taken as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group of the space-time. Equivalently, we should make the identifications on the coordinates (ψ, ϕ) to ensure regularity:

$$(\psi, \phi) \to (\psi, \phi + 2\pi) \,, \qquad (\psi, \phi) \to \left(\psi + \frac{2\pi}{\kappa_E}, \phi + \frac{2\pi\Omega_E}{\kappa_E}\right). \tag{6.16}$$

We recover a static black hole on the Euclidean Schwarzschild instanton [35] from the above solution by setting $\alpha = 0$. The background space-time, obtained by setting $c = 0$ so that the time-like rod vanishes, is nothing but the Euclidean Kerr instanton with a flat time dimension. The following redefinition of parameters and coordinate transformations

$$m = \frac{\varkappa^2 (1 - \alpha^2)}{2(1 + \alpha^2)} \,, \quad a = \frac{\alpha \varkappa^2}{1 + \alpha^2} \,, \quad r = \frac{\varkappa^2 [1 - y + \alpha^2 (1 - x)]}{(1 + \alpha^2)(x - y)} \,, \quad \cos\theta = \frac{2 + x + y}{x - y} \,, \tag{6.17}$$

are needed to bring the Euclidean Kerr instanton in the background space-time to the standard form (5.17). We thus naturally interpret the above solution as a static black hole sitting at the first turning point of the Euclidean Kerr instanton.

We can put the rod structure in standard orientation by taking $\{\tilde{V}_{(0)} = \frac{\partial}{\partial t}, \tilde{V}_{(1)} = \ell_3, \tilde{V}_{(2)} = \ell_1\}$. The corresponding new Weyl–Papapetrou coordinates $(\tilde{t}, \tilde{\psi}, \tilde{\phi}, \tilde{\rho}, \tilde{z})$ are related to the old coordinates (6.13) by:

$$t = \tilde{t} \,, \qquad \psi = \frac{1}{\kappa_E} \tilde{\psi} \,, \qquad \phi = \frac{\Omega_E}{\kappa_E} \tilde{\psi} + \tilde{\phi} \,, \qquad \rho = \kappa_E \tilde{\rho} \,, \qquad z = \kappa_E \tilde{z} \,. \tag{6.18}$$

The three turning points are now pushed to $(\tilde{\rho} = 0, \tilde{z} = \tilde{z}_i \equiv \frac{z_i}{\kappa_E})$ for $i = 1, 2, 3$, and the corresponding directions of the four rods from left to right are $K_1 = (0, 0, 1)$, $K_2 = \frac{1}{\kappa}(1, 0, 0)$, $K_3 = (0, 1, 0)$ and $K_4 = (0, 0, 1)$. This is illustrated in Fig. 6.2(a). In the new Weyl–Papapetrou coordinates, the following identifications are made to ensure regularity:

$$(\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi}, \tilde{\phi} + 2\pi) \,, \qquad (\tilde{\psi}, \tilde{\phi}) \to (\tilde{\psi} + 2\pi, \tilde{\phi}) \,. \tag{6.19}$$

As in the static black hole on the self-dual Taub-NUT instanton, no degeneracy of the rod structure can be found except in the background space-time.

It can be checked that this space-time is free of curvature singularities and CTCs, and is thus completely regular. It is obvious that the black hole has horizon topology S^3. At infinity $(x, y) \to (-1, -1)$, the space-time approaches the direct product of a flat time dimension and the asymptotic structure of the Euclidean Kerr instanton (note that Ω_E and κ_E now take the new values). Similarly as in the case of the Euclidean Kerr instanton, if $\frac{\Omega_E}{\kappa_E}$ is a rational number, the Killing vector field $\frac{\partial}{\partial \psi}$ generates a closed and finite dimension at infinity.

We expect that angular momenta can be added to the above black hole solution. Also another black hole may be added to the second turning point of the Euclidean Kerr instanton, resulting in a double black hole solution.

6.6 Black holes on the Eguchi–Hanson instanton

One may expect that, similarly as done in the last section, black holes can be placed at the turning points of the Eguchi–Hanson instanton. However, no static black hole solutions of such a class have been found, despite all the efforts we have made. It is not clear to us whether it is because of the limited solution generating techniques we have applied, or that these configurations do not exist at all.

It turns out, however, that a rotating black hole on the Eguchi–Hanson instanton does exist, and its local metric can take the same form as the single-rotating black lens solution (4.21) but with parameters a and b fixed as

$$a = \frac{3(1-c)}{3+5c}\,, \qquad b = \frac{4c(3-c)}{5c^2-6c+9}\,, \tag{6.20}$$

which are determined to solve the conditions $n = 1$ and $m = 2$ in (4.26). These conditions ensure that the black hole has horizon topology $L(1,2) \cong S^3$, and that the finite space-like rod has the correct orientation $(0, \frac{1}{2}, \frac{1}{2})$ required of an Eguchi–Hanson instanton background (subsection 5.3.5). It can be easily checked that $0 < a \leq \sqrt{(1-b)/(1+b)}$, so this solution is in Range I as defined in section 4.3. The rod structure of this solution is illustrated in Fig. 6.2(b). To ensure regularity, we identify the direction pair $\{\ell_3, \ell_4\}$ of adjacent space-like rods with period 2π independently, which can then be taken as the pair of independent 2π-periodic generators of the $U(1) \times U(1)$ isometry group of the space-time. Equivalently, the following identifications on the coordinates (ψ, ϕ) should be made to ensure regularity:

$$(\psi, \phi) \to (\psi + 2\pi, \phi)\,, \qquad (\psi, \phi) \to (\psi + \pi, \phi + \pi)\,. \tag{6.21}$$

This space-time is free of curvature singularities, and no CTCs have been found despite extensive numerical checks. At infinity $(x, y) \to (-1, -1)$ the space-time approaches a five-dimensional Minkowski space-time quotiented by a $\mathbb{Z}_2$ group, i.e., $M^{1,4}/\mathbb{Z}_2$. The background space-time, obtained by setting $c = 0$ so that the time-like rod vanishes, is nothing but the Eguchi–Hanson instanton with a flat time dimension. The following parameter redefinition and coordinate transformations

$$\bar{a} = \varkappa\,, \quad r^2 = \frac{\varkappa^2(2 - x - y)}{x - y}\,, \quad \cos\theta = \frac{2 + x + y}{x - y}\,, \quad \bar{\psi} = \psi + \phi\,, \quad \bar{\phi} = -\psi + \phi\,, \tag{6.22}$$

are needed to bring the Eguchi–Hanson solution in the background space-time to the standard form:

$$\mathrm{d}s^2 = \left(1 - \frac{\bar{a}^4}{r^4}\right)\frac{r^2}{4}\left(\mathrm{d}\bar{\psi} + \cos\theta\,\mathrm{d}\bar{\phi}\right)^2 + \left(1 - \frac{\bar{a}^4}{r^4}\right)^{-1}\mathrm{d}r^2 + \frac{r^2}{4}(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\bar{\phi}^2)\,. \tag{6.23}$$

We thus naturally interpret the above solution as a rotating black hole sitting at the first turning point of the Eguchi–Hanson instanton.

It is instructive to see what happens if we instead begin with the static black lens solution (4.1). To ensure that $n = 1$, we have to set $a = \frac{1-c}{1+c}$, in which case $m = \frac{2}{\sqrt{1-c^2}} \geq 2$. Thus, the background space-time of an Eguchi–Hanson instanton with a flat time dimension is only recovered in the limit $c = 0$, when the time-like rod vanishes and there is no black hole present. In general, when there is a black hole present, the finite space-like rod does not have the correct direction $(0, \frac{1}{2}, \frac{1}{2})$ required of an Eguchi–Hanson instanton background. Instead, this solution is more naturally interpreted as a black hole sitting on the possible new gravitational instanton conjectured in the section 5.4.1, if we set $m \equiv p \geq 3$ to be an integer so that the finite space-like rod has orientation $(0, \frac{1}{p}, \frac{1}{p})$. By identifying $\{\ell_3, \ell_4\}$ with period 2π independently, we get a black hole with horizon topology $L(1,p) \cong S^3$ in an asymptotically lens-space $L(p,1)$ space-time [82].[4]

A more general class of rotating black holes on the Eguchi–Hanson instanton may similarly be obtained from the double-rotating black lens solution that we mentioned in section 4.5. More complicated configurations, such as black rings and multi-black holes on the Eguchi–Hanson instanton, may also exist, but have not been found yet.

Black holes may also possibly be constructed on the double-centered (or even multi-collinearly-centered) Taub-NUT instantons, but to the best of our knowledge no such examples have been found in the current literature.

6.7 Black holes on the Taub-bolt instanton

The solution describing a static black hole sitting on the Euclidean non-self-dual Taub-NUT solution has a metric in the following form in C-metric coordinates:[5]

$$\mathrm{d}s^2 \;=\; -\frac{1+cy}{1+cx}\,\mathrm{d}t^2 + \frac{F(x,y)}{H(x,y)}\,(\mathrm{d}\psi+\Omega)^2$$

[4]Since c is fixed in terms of p, the background limit of this space-time is taken as $\varkappa \to 0$. In this limit, the finite space-like rod also vanishes, and we recover a quotiented five-dimensional Minkowski space-time $M^{1,4}/\mathbb{Z}_p$ (which contains an orbifold singularity and is thus singular). This suggests that the conjectured new gravitational instanton cannot exist alone as a background space; a black hole must be present in such a space-time through some unknown mechanism.

[5]This solution can be obtained from (5.20) in Ford et al. [62] by imposing $\tilde{Q} = \infty$, so that v in (5.26) has only one non-vanishing second component. Redefinition of parameters and coordinates are needed to bring the solution to exactly the same form as above.

$$+\frac{2\varkappa^4\,(1-c)\,(1+cx)\,H(x,y)}{(1-\alpha^2)\,(x-y)^3}\left(\frac{\mathrm{d}x^2}{G(x)}-\frac{\mathrm{d}y^2}{G(y)}+A\,\mathrm{d}\phi^2\right), \quad (6.24)$$

where Ω and A are given by

$$\begin{aligned}\Omega &= \frac{2\alpha\,\varkappa^2[2+x+y+c\,(1+x)\,(1+y)]}{(1-\alpha^2)\,(x-y)}\,\mathrm{d}\phi\,,\\ A &= -\frac{2\,(1+x)\,(1+y)}{(1-c)\,(x-y)}\,. \end{aligned} \quad (6.25)$$

The functions $G(x)$, $H(x,y)$ and $F(x,y)$ are defined as

$$\begin{aligned} G(x) &= (1+cx)\,(1-x^2)\,,\\ H(x,y) &= (1+cx)\,(1-y)^2-\alpha^2\,(1+cy)\,(1-x)^2\,,\\ F(x,y) &= (1-\alpha^2)\,(1-x)\,(1-y)\,(1+cx)\,. \end{aligned} \quad (6.26)$$

The parameters $\varkappa$, c, α and coordinates t, x, y take the range $\varkappa>0$, $0\le c<1$, $\alpha^2<1$, $-\infty<t<\infty$, $-1\le x\le 1$ and $-\frac{1}{c}\le y\le -1$.

The Weyl–Papapetrou coordinates (t,ψ,ϕ,ρ,z) are related to the above C-metric coordinates by the relation (6.13). In these coordinates, the locations of the three turning points in the rod structure are the same as those in the static black hole on the Euclidean Kerr instanton in section 6.5. They divide the z-axis into four rods:

- Rod 1: a semi-infinite space-like rod located at $(\rho=0, z\le z_1)$ or $(x=-1, -\frac{1}{c}\le y<-1)$, with direction $\ell_1=(0,\frac{2\alpha\varkappa^2}{1-\alpha^2},1)$.
- Rod 2: a finite time-like rod located at $(\rho=0, z_1\le z\le z_2)$ or $(-1\le x\le 1, y=-\frac{1}{c})$, with direction $\ell_2=\frac{1}{\kappa_{TN}}(1,0,0)$, where
$$\kappa_{TN}=\frac{\sqrt{2c\,(1+c)\,(1-\alpha^2)}}{4c\varkappa^2\,(1+c)}\,, \quad (6.27)$$
is the surface gravity on the horizon.

- Rod 3: a finite space-like rod located at $(\rho = 0, z_2 \leq z \leq z_3)$ or $(x = 1, -\frac{1}{c} \leq y \leq -1)$, with direction $\ell_3 = (0, \frac{2\varkappa^2\sqrt{1-c^2}}{1-\alpha^2}, 0)$.

- Rod 4: a semi-infinite space-like rod located at $(\rho = 0, z \geq z_3)$ or $(-1 < x \leq 1, y = -1)$, with direction $\ell_4 = (0, -\frac{2\alpha\varkappa^2}{1-\alpha^2}, 1)$.

The second rod is time-like and represents the black hole horizon. By setting $\alpha = 0$ we directly recover a static black hole on the Euclidean Schwarzschild instanton from the above solution. The background space-time, obtained by setting $c = 0$ so that the time-like rod vanishes, is nothing but the non-self-dual Taub-NUT solution with a flat time dimension. The following redefinition of parameters and coordinate transformations

$$m = \frac{\varkappa^2\,(1+\alpha^2)}{2(1-\alpha^2)}\,, \quad n = \frac{\alpha\varkappa^2}{1-\alpha^2}\,, \quad r = \frac{\varkappa^2[1-y-\alpha^2\,(1-x)]}{(1-\alpha^2)\,(x-y)}\,, \quad \cos\theta = \frac{2+x+y}{x-y}\,, \tag{6.28}$$

are needed to bring the non-self-dual Taub-NUT solution in the background space-time to the standard form (5.35).

Similarly as in the case of the non-self-dual Taub-NUT solution, the solution (6.24), in general, does not describe a regular space-time. A regular class can be obtained by fixing $\alpha = \pm\frac{\sqrt{1-c^2}}{2}$, to which we pay attention from now on. In this case, the rod directions in the rod structure now become $\ell_1 = (0, 2n, 1)$, $\ell_2 = \frac{1}{\kappa_{TB}}(1, 0, 0)$, $\ell_3 = (0, 4n, 0)$ and $\ell_4 = (0, -2n, 1)$, with NUT charge $n = \pm\frac{2\varkappa^2\sqrt{1-c^2}}{3+c^2}$, and surface gravity on the horizon $\kappa_{TB} = \frac{\sqrt{2c(1+c)(3+c^2)}}{8c\varkappa^2(1+c)}$, so now ℓ_1 can be expressed as an integral linear combination of ℓ_3 and ℓ_4. To ensure regularity, we identify the direction pair $\{\ell_3, \ell_4\}$ of adjacent space-like rods with period 2π independently, which can then be taken as the pair of independent 2π-periodic generators of the $U(1)\times U(1)$ isometry group of the space-time. Equivalently, the following identifications on the coordinates (ψ,ϕ) should be made to ensure regularity:

$$(\psi,\phi)\to(\psi+8n\pi,\phi)\,,\qquad(\psi,\phi)\to(\psi-4n\pi,\phi+2\pi)\,.\tag{6.29}$$

The background space-time of this regular class of solutions is nothing but the Taub-bolt instanton with a flat time dimension. We thus naturally interpret this solution as a static black hole sitting at the first turning point of the Taub-bolt instanton.

We can put the rod structure in standard orientation by taking $\{\tilde{V}_{(0)}=\frac{\partial}{\partial t},\tilde{V}_{(1)}=\ell_4,\tilde{V}_{(2)}=\ell_1\}$. The corresponding new Weyl–Papapetrou coordinates $(\tilde{t},\tilde{\psi},\tilde{\phi},\tilde{\rho},\tilde{z})$ are related to the old coordinates (6.13) by:

$$t=\tilde{t}\,,\quad\psi=2n(\tilde{\psi}+\tilde{\phi})\,,\quad\phi=-\tilde{\psi}+\tilde{\phi}\,,\quad\rho=\frac{1}{4|n|}\tilde{\rho}\,,\quad z=\frac{1}{4|n|}\tilde{z}\,.\tag{6.30}$$

The three turning points are now pushed to $(\tilde{\rho}=0,\tilde{z}=\tilde{z}_i\equiv4|n|z_i)$ for $i=1,2,3$, and the corresponding directions of the four rods from left to right are $K_1=(0,0,1)$, $K_2=\frac{1}{\kappa_{TB}}(1,0,0)$, $K_3=(0,1,1)$ and $K_4=(0,1,0)$. This is illustrated in Fig. 6.2(c). In the new Weyl–Papapetrou coordinates, the following identifications are made to ensure regularity:

$$(\tilde{\psi},\tilde{\phi})\to(\tilde{\psi},\tilde{\phi}+2\pi)\,,\qquad(\tilde{\psi},\tilde{\phi})\to(\tilde{\psi}+2\pi,\tilde{\phi})\,.\tag{6.31}$$

Again, no degeneracy of the rod structure can be found.

It can be checked that this space-time is free of curvature singularities and CTCs, and is thus completely regular. The black hole has horizon topology S^3. At infinity $(x,y)\to(-1,-1)$, the space-time approaches the direct product of a flat time dimension and the asymptotic structure of the Taub-bolt instanton with NUT charge n.

We expect that angular momenta can be added to the above black hole solution. Also another black hole may be added to the second turning point of the Taub-bolt instanton, resulting in a double black hole solution.

6.8 Discussion

In this chapter we have classified and constructed black holes on gravitational instantons with $U(1) \times U(1)$ isometry and up to two turning points (except the double-centered Taub-NUT instanton). These black holes are completely regular five-dimensional space-times and possess an isometry group $\mathbb{R} \times U(1)^2$. Most of the known exact five-dimensional black holes in the literature have been classified within our scheme, and some new classes of solutions have been constructed as black holes on the Euclidean Kerr, Eguchi–Hanson and Taub-bolt instantons respectively.

It is known that five-dimensional black hole space-times that asymptotically approach $M^{1,4}$ or $M^{1,3} \times S^1$ have very rich structures [20–22]. The black holes/rings that have been classified or constructed in this chapter have various and more general asymptotic geometries. In such space-times, we also expect very rich black hole structures. A possible extension of the work in this chapter is then to construct the most general double-rotating black holes/rings, or their superpositions, on gravitational instantons; in particular, the explicit forms of the double-rotating black ring on Taub-NUT, and the double-rotating black holes on Euclidean Kerr and

Taub-bolt. Such constructions will be helpful to understand the phase structure of these space-times. It is also worthwhile to examine and extend the black hole mechanics/thermodynamics to space-times with the new asymptotic geometries considered in this chapter. Appropriate physical quantities of these space-times might be identified and calculated, which can be used to characterize these solutions and prove their (non-)uniqueness theorems.

We note that black holes placed at a turning point (thus covering it) of a gravitational instanton will have a horizon topology S^3; while those placed somewhere on a space-like rod (thus covering part of it) will have a horizon topology $S^1 \times S^2$. Thus the classification of black holes in five dimensions with $\mathbb{R} \times U(1)^2$ isometry as black holes/rings on gravitational instantons is not a complete, though practical and useful, scheme. For example, if a completely regular black lens exists [38, 74, 96], it cannot be classified as a black hole on a gravitational instanton in general, since its event horizon has a topology of a lens-space $L(n, 1)$, which is a $\mathbb{Z}_n$ quotient of S^3.

It will be interesting to seek possible interpretations in Kaluza–Klein theory of the black hole solutions on AF/ALF gravitational instantons presented in this chapter. As already mentioned, if a trivial flat time dimension is added to an AF/ALF gravitational instanton, the resulting solution has a description in Kaluza–Klein theory of a magnetic monopole or dipole, or their superpositions [125, 126]. We expect that the solutions of black holes on AF/ALF gravitational instantons presented in this chapter may be interpreted in Kaluza–Klein theory as magnetically and/or electrically charged black holes, possibly in superposition with monopoles or dipoles. We note that, however, when a five-dimensional space-time with $\mathbb{R} \times U(1)^2$

isometry is dimensionally reduced along the direction of say the Killing vector field $\frac{\partial}{\partial\psi}$, in Kaluza–Klein theory, the resulting system will consist of a stationary space-time with $\mathbb{R} \times U(1)$ isometry, say parameterized by t and ϕ, a gauge field and a scalar field. This means that the identifications that are made in this chapter for the five-dimensional space-time to be regular, are not necessary in Kaluza–Klein theory; though in certain cases, these identifications may play a relevant role in the latter theory. So it is possible that even if we start from a completely regular five-dimensional space-time, singular objects might appear in the dimensionally reduced system in Kaluza–Klein theory. In the next chapter, we will consider the relations between black holes on the Taub-NUT instanton, which is the simplest ALF gtavitational instanton, and Kaluza–Klein black holes.

Black holes on certain gravitational instantons in other contexts, such as five-dimensional Einstein–Maxwell theory and minimal supergravity, might also be constructed. We already mentioned at the beginning of this chapter some recent progress made in this direction in the literature.

Chapter 7

Black holes on Taub-NUT and Kaluza–Klein black holes

7.1 Introduction

We already mentioned in section 6.3 that, black holes on Taub-NUT as vacuum solutions to five-dimensional gravity, when dimensionally reduced along the compact direction, have descriptions as magnetically charged black holes in Kaluza–Klein (KK) theory. In this chapter we elaborate the details of the intimate connections between these black holes. The material presented in this chapter is based on our unpublished paper [77], the work of which partially overlaps with that of Emparan et al. [137, 138].

First we briefly review here the Kaluza–Klein theory. The original idea of Kaluza and Klein [140, 141] is to unify gravity and electromagnetism in terms of five-dimensional pure gravity, compactified along an extra and very small dimension. This idea of unifications by means of extra dimensions and dimensional reduction has long been one of the most fascinating and recurring subjects of modern physics.

Pure gravity in five dimensions with metric ${}^{(5)}g_{\mu\nu}$ has the Hilbert–Einstein action

$$S = \int \mathrm{d}^5x \sqrt{-{}^{(5)}g}\, {}^{(5)}R, \tag{7.1}$$

from which the well-known (vacuum) Einstein equations can be derived. Here ${}^{(5)}g$ and ${}^{(5)}R$ are the determinant and Ricci scalar of the five-dimensional metric ${}^{(5)}g_{\mu\nu}$ respectively. If we assume the space-time is independent of the fifth dimension parameterized by x^5, we can then employ the following ansatz of the five-dimensional metric

$$\mathrm{d}s^2 = e^{4\sigma/\sqrt{3}} \left(\mathrm{d}x^5 + 2A_\mu \mathrm{d}x^\mu\right)^2 + e^{-2\sigma/\sqrt{3}} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{7.2}$$

where the indices μ, ν run over from 0 to 3. Substituting this metric into (7.1), the action now reduces to

$$S = \int \mathrm{d}x^5 \int \mathrm{d}^4x \sqrt{-g} \left[R - 2(\partial\sigma)^2 - e^{2\sigma\sqrt{3}} F^2\right], \tag{7.3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The functions g and R are the determinant and Ricci scalar of the four-dimensional metric $g_{\mu\nu}$ respectively; index lowering and raising in (7.3) are performed also with respect to this four-dimensional metric. Since the second integral in (7.3) does not involve x^5, the system (7.3) now can be seen as four-dimensional gravity $g_{\mu\nu}$ coupled to a gauge field A_μ and a scalar field σ. In fact, such a system can be embedded as an Einstein–Maxwell–dilaton theory with coupling constant $\lambda = \sqrt{3}$. The equations of motion for such a system (7.3) are now

$$\begin{aligned} R_{\mu\nu} &= 2(\partial_\mu\sigma)(\partial_\nu\sigma) + 2e^{2\sqrt{3}\sigma}T_{\mu\nu}, \\ \nabla_\mu(e^{2\sqrt{3}\sigma}F^{\mu\nu}) &= 0, \\ \Box\sigma &= \tfrac{\sqrt{3}}{2}e^{2\sqrt{3}\sigma}F^{\mu\nu}, \end{aligned} \tag{7.4}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the gauge field A_μ given by

$$T_{\mu\nu} = F_{\mu\alpha}F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^2. \tag{7.5}$$

Once we have a system ($g_{\mu\nu}$, A_ν, σ) satisfying the above equations, we can always lift it to a five-dimensional metric via (7.2), as a solution of the vacuum Einstein equations. Conversely, we can also start from a five-dimensional metric that is independent of the fifth dimension, and dimensionally reduce it to the system ($g_{\mu\nu}$, A_ν, σ) satisfying (7.4). It is worthwhile to remark that, because of the non-vanishing coupling in the system (7.4), the dilaton field σ plays a non-trivial role. It is easy to see that σ can be set to constant only in the case when $F^2 = 0$, which, however, does not hold in general in the Einstein–Maxwell theory.

It turns out that the Kaluza–Klein theory (7.3) admits very rich solutions. In particular, it admits magnetic monopole solutions [125, 126]. In fact, these solutions are obtained by adding a flat time dimension to the Taub-NUT instanton, and performing dimensional reduction. Magnetic dipoles, or their superpositions with monopoles can also be obtained by performing dimensional reduction on other AF/ALF gravitational instantons. It is also possible to find black hole solutions in such a theory. The static dyonic Kaluza–Klein black holes were found in [130–132], and thoroughly analyzed by Gibbons and Wiltshire [133]. They were generalized by Rasheed [91], Matos and Mora [134], and Larsen [135] to the rotating dyonic solutions.

In recent years, five-dimensional black hole space-times have been constructed using the Taub-NUT instanton as background spaces. In particular, a vacuum solution was found by Ishihara and Matsuno [60], which can be interpreted as a Schwarzschild black hole on Taub-NUT. This solution was subsequently generalized by Wang [61] to a Myers–Perry (MP) black hole with equal momenta on Taub-NUT. These black holes are considered as completely regular five-dimensional space-times, as vacuum solutions to Einstein field equations. These black hole solutions exhibit an isometry group $\mathbb{R} \times U(1)^2$, and can be analyzed in terms of rod structure. We have already analyzed the Ishihara–Matsuno solution and interpreted it as the Schwarzschild black hole on Taub-NUT in section 6.3. Similar analysis can be done for Wang's solution. It is expected that Wang's solution can be generalized to a double-rotating Myers–Perry black hole on Taub-NUT. Since all these black holes are five-dimensional vacuum solutions, it is natural to ponder what are their descriptions in Kaluza–Klein theory via dimensional reduction, as just reviewed.

It is the purpose of this chapter to address the intimate relations between black holes on Taub-NUT and Kaluza–Klein black holes. In section 7.5, the most general rotating dyonic Kaluza–Klein black hole, when appropriately lifted to five dimensions, is identified with the double-rotating Myers–Perry black hole on Taub-NUT. We also discuss in detail some of its special cases. In section 7.2, we show that the Ishihara–Matsuno solution [60], when dimensionally reduced to four dimensions, can be identified with the static magnetically charged Kaluza–Klein black hole. In section 7.3, we show that Wang's solution [61] describes the static dyonic Kaluza–Klein black hole when dimensionally reduced to four dimensions. We present a new solution in section 7.4 and interpret it as a Myers–Perry black hole with opposite angular momenta on Taub-NUT; when dimensionally reduced to four dimensions, it describes a rotating magnetically charged Kaluza–Klein black hole. We conclude this chapter with a discussion in section 7.6.

We remark that, although black holes on Taub-NUT and Kaluza–Klein black holes are intimately connected, and can be obtained from the same five-dimensional metric, their interpretations are essentially different. Black holes on Taub-NUT are completely regular five-dimensional space-times with isometry group $\mathbb{R} \times U(1)^2$, so we should impose certain identifications on the coordinates as analyzed in subsection 3.1.2. However, in this chapter, Kaluza–Klein black holes are taken as essentially four-dimensional, with an isometry group $\mathbb{R} \times U(1)$; their regularity conditions are quite different from those of black holes on Taub-NUT. As a consequence, we note that, a completely regular solution of a black hole on Taub-NUT, when dimensionally reduced to four dimensions, may not always possibly describe a regular Kaluza–Klein black hole.

7.2 Schwarzschild black hole on Taub-NUT & static magnetic KK black hole

The solution describing a static black hole on Taub-NUT was discovered by Ishihara and Matsuno [60]. We have analysed this solution in section 6.3 and interpreted it as the Schwarzschild black hole on Taub-NUT. Here, for clarity of presentation, we write down its metric (6.4) again

$$\mathrm{d}s^2 = \frac{r^2}{4}\left(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi\right)^2 - \left(1 - \frac{2m}{r^2}\right)\mathrm{d}t^2 + \frac{k(r)^2}{1-\frac{2m}{r^2}}\mathrm{d}r^2 + \frac{k(r)r^2}{4}\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{7.6}$$

where the function $k(r)$ is defined as

$$k(r) = \frac{b^2(b^2-2m)}{(b^2-r^2)^2}. \tag{7.7}$$

The coordinate r takes the range $\sqrt{2m} \leq r \leq b$, with the lower and upper bounds corresponding to the black hole horizon and physical infinity respectively. When the mass parameter $m = 0$, we recover the background of the self-dual Taub-NUT instanton (with NUT charge $\frac{b}{4}$) with a flat time dimension.

We notice that, in (7.6), the closer we move towards the event horizon at $r = \sqrt{2m}$, the closer the function $k(r)$ becomes to unity, and thus the more the space-time behaves like the five-dimensional Schwarzschild black hole. On the other hand, when we move toward the physical infinity $r \to b$, the function $k(r)$ can be very large, and the space-time can be then very different from being asymptotically $M^{1,4}$; in fact, the function $k(r)$ turns the asymptotic structure to a finite S^1 fiber bundle over $M^{1,3}$. Thus as we observe from far away, the black hole (7.6) becomes effectively four-dimensional; while as we observe close to the horizon, it becomes more like five-dimensional. This behavior was first observed in [142]. Also, thanks to the extra factor $k(r)$, the event horizon deviates from a perfect around S^3, and looks "squashed" [60]. These behaviors are shared by the double-rotating black hole on Taub-NUT.

Recall that in Kaluza–Klein theory, after dimensional reduction, the self-dual Taub-NUT instanton with a flat time dimension describes a magnetic monopole. One then naturally expects that, after dimensional reduction, the above solution (7.6) would describe a static magnetically charged black hole in Kaluza–Klein theory. In fact, the latter solution was found long ago, and was contained as a special case in the most general rotating dyonic Kaluza–Klein black hole [91, 134, 135]. This general solution, with a form (7.26), will be discussed in detail later. By setting $(a = 0, q = 2n)$, we recover from (7.26) the static magnetically charged Kaluza–Klein black hole; if lifted into five dimensions, it has the following form

$$\begin{aligned} \mathrm{d}s^2 &= \frac{R^2}{R^2+(p-2n)R}\left[\mathrm{d}y^2+\sqrt{p(p-2n)}\cos\Theta\,\mathrm{d}\Phi\right]^2-\frac{R^2-2nR}{R^2}\mathrm{d}T^2 \\ &\quad+\left[R^2+(p-2n)R\right]\left(\frac{\mathrm{d}R^2}{R^2-2nR}+\mathrm{d}\Theta^2+\sin^2\Theta\,\mathrm{d}\Phi^2\right). \end{aligned} \tag{7.8}$$

The extra dimension is parameterized by y; the four-dimensional metric $g_{\mu\nu}$, gauge field A_μ and dilaton field ϕ in Kaluza–Klein theory can be easily read off from the above solution by comparing it with the ansatz (7.2). It is not difficult to see that the four-dimensional metric is static, and that the gauge field carries a pure magnetic charge. We find that this solution (7.8) is indeed equivalent to the static black hole on Taub-NUT (7.6) by the following coordinate transformations and redefinitions of parameters

$$\begin{gathered} R=\tfrac{\sqrt{b^2-2m}}{2(b^2-r^2)}r^2, \qquad T=\tfrac{\sqrt{b^2-2m}}{b}t, \qquad y=\tfrac{b}{2}\psi, \qquad \Theta=\theta, \qquad \Phi=\phi, \\ p=\tfrac{b^2}{2\sqrt{b^2-2m}}, \qquad n=\tfrac{m}{2\sqrt{b^2-2m}}. \end{gathered} \tag{7.9}$$

This confirms the expectation that the solution (7.6), after dimensional reduction, describes a magnetically charged Kaluza–Klein black hole, with the magnetic charge arising from the Taub-NUT background, and with the Kaluza–Klein black hole arising from the black hole in five-dimensional pure gravity.

7.3 MP black hole with $a_1 = a_2$ on Taub-NUT & static dyonic KK black hole

The Schwarzschild black hole on Taub-NUT of Ishihara and Matsuno (7.6) was obtained by first writing the five-dimensional Schwarzschild solution in terms of the left-invariant one-forms (σ_1, σ_2, σ_3) on S^3, which naturally defines the Hopf fibration of S^3 as S^1 fiber bundle over the base space S^2, and then introducing certain extra factors to the fiber S^1 and base space S^2 respectively [60]. A similar procedure was subsequently used by Wang to construct his rotating Kaluza–Klein black hole with squashed horizon [61]. The starting point of Wang's construction is the five-dimensional Myers–Perry black hole with equal angular momenta. Then to apply the above-mentioned procedure, Wang rewrote it in a new form in terms of the left-invariant one-forms, as done in the following.

The five-dimensional Myers–Perry black hole with two independent angular momenta, as reviewed in section 2.2, has the usual form

$$\begin{aligned} \mathrm{d}s^2 \;&=\; \frac{2m}{\Sigma}\left(\mathrm{d}t - a_1 \sin^2\theta\, \mathrm{d}\phi_1 - a_2 \cos^2\theta\, \mathrm{d}\phi_2\right)^2 - \mathrm{d}t^2 + \Sigma\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right) \\ &\quad + \left(r^2 + a_1^2\right)\sin^2\theta\, \mathrm{d}\phi_1^2 + \left(r^2 + a_2^2\right)\cos^2\theta\, \mathrm{d}\phi_2^2\,, \end{aligned} \tag{7.10}$$

where the functions Σ and Δ are given by

$$\Sigma = r^2 + a_1^2\cos^2\theta + a_2^2\sin^2\theta, \qquad \Delta = \frac{(r^2 + a_1^2)\,(r^2 + a_2^2)}{r^2} - 2m. \tag{7.11}$$

ϕ_1 and ϕ_2 parameterize the two asymptotic axes, and a_1 and a_2 are the corresponding angular momentum parameters along these axes. Setting $a_1 = a_2 = a$ we obtain the five-dimensional Myers–Perry black hole with equal angular momenta. If we further perform the coordinate transformations ($\phi_1 \to \frac{\psi-\phi}{2}, \phi_2 \to \frac{\psi+\phi}{2}, \theta \to 2\theta$), we can write the solution in the following form

$$\mathrm{d}s^2 = \frac{h_2}{4h_1}\left(\mathrm{d}\psi - \frac{4ma}{h_2}\,\mathrm{d}t + \cos\theta\,\mathrm{d}\phi\right)^2 - \frac{h_3}{h_2}\,\mathrm{d}t^2 + \frac{h_1}{4}\left(\frac{4r^2}{h_3}\,\mathrm{d}r^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{7.12}$$

where the functions h_1, h_2 and h_3 are defined as

$$h_1 = r^2 + a^2, \qquad h_2 = \left(r^2 + a^2\right)^2 + 2ma^2, \qquad h_3 = \left(r^2 + a^2\right)^2 - 2mr^2. \tag{7.13}$$

This form of the solution can be obtained from (7.34) by setting $a_1 = a_2 = a$.

Wang's rotating Kaluza–Klein black hole [61] can be then obtained by introducing extra factors to the above solution (7.12) in the following form[1]

$$\mathrm{d}s^2 = \frac{h_2}{4h_1}\left(\mathrm{d}\psi - \frac{4ma}{h_2}\,\mathrm{d}t + \cos\theta\,\mathrm{d}\phi\right)^2 - \frac{h_3}{h_2}\,\mathrm{d}t^2 + \frac{k(r)h_1}{4}\left[\frac{4k(r)r^2}{h_3}\,\mathrm{d}r^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right], \tag{7.14}$$

with the extra factor $k(r)$ given by

$$k(r) = \frac{\left(a^2 + b^2\right)^2 - 2mb^2}{\left(b^2 - r^2\right)^2}. \tag{7.15}$$

The coordinate r takes the range $\sqrt{m - a^2 + \sqrt{m(m - 2a^2)}} \leq r \leq b$, with the lower and upper bounds corresponding to the black hole horizon and physical infinity respectively. Notice that the introduction of the extra factor alters the ranges of coordinates. When $b \to \infty$, the extra factor $k(r) \to 1$, and then we recover the five-dimensional Myers–Perry black hole with equal angular momenta (7.12). The parameters m, a satisfy the inequality $m \geq 2a^2$. If we take the limit $m \to 0$ while respecting this inequality, we recover the self-dual Taub-NUT instanton with a flat

[1]This form of solution (7.14) can be obtained from Eq. (9) in [61]. To match exactly, we need to do the substitution $m \to 2m$ and some obvious relabeling of parameters for Eq. (9) in [61].

time dimension. Furthermore, the rod structure of this solution can be calculated; it can be shown that it is qualitatively the same as that of (7.6) except that the horizon now has non-vanishing angular velocities. This black hole has an event horizon with a topology of S^3, as that of (7.6), if appropriate identifications on the coordinates ψ and ϕ are made. When $a = 0$, we recover the Schwarzschild black hole on Taub-NUT (7.6). Thus it is natural to interpret the above solution (7.14) as the (5D) Myers–Perry black hole with equal angular momenta on Taub-NUT.

It is not difficult to see that the solution (7.14), after dimensional reduction along ψ, would describe a static dyonic Kaluza–Klein black hole. This is indeed true, as to be proved. The static dyonic Kaluza–Klein black hole [133], obtained from the most general Kaluza–Klein black hole (7.26) by setting $a = 0$, when lifted to five dimensions, has the following form

$$\begin{aligned} \mathrm{d}s^2 \;&=\; \frac{H_2}{H_1}\left[dy - \frac{\sqrt{\frac{q(q^2-4n^2)}{p+q}}\left(R + \frac{p-2n}{2}\right)}{H_2}\,\mathrm{d}T + \sqrt{\frac{p(p^2-4n^2)}{(p+q)}}\cos\Theta\,\mathrm{d}\Phi\right]^2 - \frac{H_3}{H_2}\,\mathrm{d}T^2 \\ &\quad + H_1\left(\frac{\mathrm{d}R^2}{H_3} + \mathrm{d}\Theta^2 + \sin^2\Theta\,\mathrm{d}\Phi^2\right), \end{aligned} \tag{7.16}$$

where the functions H_1, H_2 and H_3 are defined as

$$\begin{aligned} H_1 &= R^2 + (p-2n)R + \frac{p(p-2n)(q-2n)}{2(p+q)}, \\ H_2 &= R^2 + (q-2n)R + \frac{q(p-2n)(q-2n)}{2(p+q)}, \\ H_3 &= R^2 - 2nR. \end{aligned} \tag{7.17}$$

After tedious algebraic calculations, it can be shown that Wang's solution (7.14) is equivalent to the above solution by the following coordinate transformations and redefinitions of parameters

$$
\begin{aligned}
R &= \sqrt{\frac{a^2+b^2}{4\left[(a^2+b^2)^2-2mb^2\right]}}\left\{\frac{(a^2+b^2)^2-2mb^2}{b^2-r^2}\right. \\
&\quad \left. -a^2-b^2+m+\sqrt{m\,(m-2a^2)}\right\}, \\
T &= \sqrt{\frac{(a^2+b^2)^2-2mb^2}{(a^2+b^2)^2+2ma^2}}t, \\
y &= \sqrt{\frac{(a^2+b^2)^2+2ma^2}{4(a^2+b^2)}}\left[\psi-\frac{4ma}{(a^2+b^2)^2+2ma^2}t\right], \\
\Theta &= \theta, \\
\Phi &= \phi, \\
p &= \frac{(a^2+b^2)^2-2ma^2}{2\,(a^2+b^2)}\sqrt{\frac{a^2+b^2}{(a^2+b^2)^2-2mb^2}}, \\
q &= \frac{m[(a^2+b^2)^2-2ma^2]}{(a^2+b^2)^2+2ma^2}\sqrt{\frac{a^2+b^2}{(a^2+b^2)^2-2mb^2}}, \\
n &= \sqrt{\frac{m\,(a^2+b^2)\,(m-2a^2)}{4[(a^2+b^2)^2-2mb^2]}}.
\end{aligned}
\tag{7.18}
$$

7.4 MP black hole with $a_1 = -a_2$ on Taub-NUT & rotating magnetic KK black hole

It is natural to wonder whether we can add several extra factors to the five-dimensional Myers–Perry black hole with opposite angular momenta to send it to the self-dual Taub-NUT background space. This indeed can be done. First we need to write the five-dimensional Myers–Perry black hole with opposite angular momenta in a new form. We set in (7.10) $a_2 = -a_1 = a$, and perform the coordinate transformations ($\phi_1 \to \frac{\psi-\phi}{2}, \phi_2 \to \frac{\psi+\phi}{2}, \theta \to 2\theta$). Then we obtain the five-dimensional Myers–Perry black hole with opposite angular momenta in the following form

$$\begin{aligned}
\mathrm{d}s^2 \;&=\; \frac{h_2}{4h_1}\left(\mathrm{d}\psi - \frac{4ma}{h_2}\cos\theta\,\mathrm{d}t + \frac{h_2+2ma^2\sin^2\theta}{h_2}\cos\theta\,\mathrm{d}\phi\right)^2 - \frac{h_3}{h_2}\Big(\mathrm{d}t+ \\
&\quad +\frac{mah_1}{h_3}\sin^2\theta\,\mathrm{d}\phi\Big)^2 + h_1\left(\frac{r^2}{\delta}\,\mathrm{d}r^2 + \frac{\mathrm{d}\theta^2}{4} + \frac{\delta}{4h_3}\sin^2\theta\,\mathrm{d}\phi^2\right), \qquad (7.19)
\end{aligned}$$

where the functions h_1, h_2, h_3 and δ are defined as

$$\begin{aligned}
h_1 &= r^2+a^2, \\
h_2 &= \left(r^2+a^2\right)^2 + 2ma^2\cos^2\theta, \\
h_3 &= \left(r^2+a^2\right)\left(r^2+a^2-2m\right) - 2ma^2\cos^2\theta, \\
\delta &= \left(r^2+a^2\right)^2 - 2mr^2. \qquad (7.20)
\end{aligned}$$

The solution (7.19) can be obtained from (7.34) by setting $a_2 = -a_1 = a$.

We then introduce two extra factors k_1 and k_2 to the above solution and obtain a new solution in the form

$$\begin{aligned}
\mathrm{d}s^2 \;&=\; \frac{h_2}{4h_1k_1}\left(\mathrm{d}\psi - \frac{4ma}{h_2}\cos\theta\,\mathrm{d}t + \frac{h_2+2ma^2\sin^2\theta}{h_2}\cos\theta\,\mathrm{d}\phi\right)^2 - \frac{h_3}{h_2}\Big(\mathrm{d}t+ \\
&\quad +\frac{mah_1}{h_3}\sin^2\theta\,\mathrm{d}\phi\Big)^2 + h_1k_2\left(\frac{r^2}{\delta}\,\mathrm{d}r^2 + \frac{\mathrm{d}\theta^2}{4} + \frac{\delta}{4h_3}\sin^2\theta\,\mathrm{d}\phi^2\right), \qquad (7.21)
\end{aligned}$$

where k_1 and k_2 are defined as

$$\begin{aligned}
k_1 &= \frac{\left(r^2+a^2\right)\left(r^2+a^2+b^2\right)+2ma^2\cos^2\theta}{b^2\left(r^2+a^2\right)}, \\
k_2 &= \frac{b^2}{b^2+2m}k_1. \qquad (7.22)
\end{aligned}$$

The coordinate r takes the range $\sqrt{m-a^2+\sqrt{m(m-2a^2)}} \le r < \infty$, with the lower and upper bounds corresponding to the black hole horizon and physical infinity respectively. Obviously the two extra factors just introduced have the property that they turn to unity when $b \to \infty$, in which case we recover the Myers–Perry black hole with opposite angular momenta (7.19). On the other hand, by taking $m \to 0$ while respecting the inequality $m \geq 2a^2$, we recover the self-dual Taub-NUT instanton with a flat time dimension. Introducing the two extra factors does not change the ranges of coordinates r and θ, but it changes the asymptotic structure from $M^{1,4}$ of (7.19) to a twisted but finite S^1 fiber bundle over $M^{1,3}$ of (7.21). The rod structure of the above solution is expected to be similar to that of the solution (7.14). It can be checked that this black hole has an event horizon with a topology of S^3, as that of (7.6), if appropriate identifications on the coordinates ψ and ϕ are made. We naturally interpret the solution (7.21) as the (5D) Myers–Perry black hole with opposite angular momenta on Taub-NUT.

What is the description of the solution (7.21) in Kaluza–Klein theory via dimensional reduction along $\frac{\partial}{\partial\psi}$? We show in the following that it is the rotating magnetic Kaluza–Klein black hole, which, when appropriately lifted to five dimensions, can be obtained from (7.26) by setting $q = 2n$ with the form

$$
\begin{aligned}
\mathrm{d}s^2 \;=\; & \frac{H_2}{H_1}\Big[\mathrm{d}y + \frac{\sqrt{2n\,(p-2n)}}{H_2}\alpha\cos\Theta\,\mathrm{d}T + \frac{\sqrt{p\,(p-2n)}\,\big(H_2 + \alpha^2\sin^2\Theta\big)}{H_2}\cos\Theta\,\mathrm{d}\Phi\Big]^2 \\
& -\frac{H_3}{H_2}\left(\mathrm{d}T - \frac{\sqrt{2pn}}{H_3}\alpha R\sin^2\Theta\,\mathrm{d}\Phi\right)^2 + H_1\left(\frac{\mathrm{d}R^2}{\Delta} + \mathrm{d}\Theta^2 + \frac{\Delta}{H_3}\sin^2\Theta\,\mathrm{d}\Phi^2\right),
\end{aligned}
\tag{7.23}
$$

where the functions H_1, H_2, H_3 and Δ are defined as

$$
\begin{aligned}
H_1 &= R^2 + (p-2n)R + \alpha^2\cos^2\Theta\,, \\
H_2 &= R^2 + \alpha^2\cos^2\Theta\,, \\
H_3 &= R^2 - 2nR + \alpha^2\cos^2\Theta\,, \\
\Delta &= R^2 - 2nR + \alpha^2.
\end{aligned}
\tag{7.24}
$$

The metric of the Myers–Perry black hole with opposite angular momenta on Taub-NUT (7.21) is equivalent to that of the rotating magnetic Kaluza–Klein black hole solution in the five-dimensional form (7.23) via the following coordinate transformations and redefinitions of parameters

$$R = \tfrac{1}{4}\left(r^2 + a^2\right), \qquad \Theta = \theta, \qquad y = \tfrac{b\sqrt{b^2+2m}}{4}\psi, \qquad T = \tfrac{\sqrt{b^2+2m}}{2}t, \qquad \Phi = \phi,$$
$$\alpha = -\sqrt{\tfrac{m}{8}}a, \qquad n = \tfrac{m}{4}, \qquad p = \tfrac{1}{4}\left(b^2 + 2m\right). \tag{7.25}$$

up to an overall factor $\frac{1}{4}(b^2+2m)$,[2] which can be simply normalized to unity by an appropriate common rescaling of the coordinates with length dimension in (7.23).

7.5 Double-rotating MP black hole on Taub-NUT & general KK black hole

The solutions (7.14) and (7.21) of black holes on Taub-NUT have a single rotational parameter; when the NUT charge characterized by b is taken to infinity, they are "blown up" to the five-dimensional Myers–Perry black hole with equal and opposite angular momenta respectively. When dimensionally reduced to four dimensions in Kaluza–Klein theory, they are subclasses of the rotating dyonic Kaluza–Klein black holes. From these connections, we are motivated to interpret the solution by appropriately lifting the rotating dyonic Kaluza–Klein black hole to five dimensions as the double-rotating Myers–Perry black hole on Taub-NUT, as generalizations of the solutions (7.14) and (7.21).

[2]This is the ratio of (7.23) to (7.21) after performing the coordinate transformations and redefinitions of parameters (7.25).

General KK black hole $\Leftrightarrow$ MP black hole on Taub-NUT

The general rotating dyonic Kaluza–Klein black hole solution, when appropriately lifted to five dimensions, in the form given by Larsen [135], can be written as[3]

$$\mathrm{d}s^2 = \frac{H_2\,(\mathrm{d}y + A_1)^2}{H_1} - \frac{H_3\,(\mathrm{d}\tau + B_1)^2}{H_2} + H_1\left(\frac{\mathrm{d}R^2}{\Delta} + \mathrm{d}\Theta^2 + \frac{\Delta}{H_3}\sin^2\Theta\,\mathrm{d}\Phi^2\right), \tag{7.26}$$

where the one-forms A_1 and A_2 are defined as

$$\begin{aligned}
A_1 \;&=\; -\left(\sqrt{\frac{q\,(q^2-4\,n^2)}{p+q}}\left(R+\frac{p-2n}{2}\right) - \sqrt{\frac{q^3\,(p^2-4\,n^2)}{4n^2(p+q)}}\,\alpha\cos\Theta\right) {H_2}^{-1}\mathrm{d}\tau \\
&\quad + \left\{\sqrt{\frac{p\,(p^2-4\,n^2)}{p+q}}\left(H_2+\alpha^2\sin^2\Theta\right)\cos\Theta - \sqrt{\frac{p\,(q^2-4\,n^2)}{4n^2(p+q)}}\times\right. \\
&\quad \left.\times\left[pR - n\,(p-2\,n) + \frac{q\,(p^2-4\,n^2)}{p+q}\right]\alpha\sin^2\Theta\right\} {H_2}^{-1}\mathrm{d}\Phi, \\
B_1 \;&=\; -\frac{\sqrt{pq}\,[(pq+4\,n^2)\,R - n\,(p-2\,n)\,(q-2\,n)]}{2n\,(p+q)\,H_3}\,\alpha\sin^2\Theta\,\mathrm{d}\Phi,
\end{aligned} \tag{7.27}$$

and the functions H_1, H_2, H_3 and Δ are given by

$$\begin{aligned}
H_1 \;&=\; R^2 + \alpha^2\cos^2\Theta + R\,(p-2\,n) + \frac{p\,(p-2\,n)\,(q-2\,n)}{2(p+q)} \\
&\quad + \frac{p}{2n\,(p+q)}\sqrt{(p^2-4\,n^2)\,(q^2-4\,n^2)}\,\alpha\cos\Theta, \\
H_2 \;&=\; R^2 + \alpha^2\cos^2\Theta + R\,(q-2\,n) + \frac{q\,(p-2\,n)\,(q-2\,n)}{2(p+q)} \\
&\quad - \frac{q}{2n\,(p+q)}\sqrt{(p^2-4\,n^2)\,(q^2-4\,n^2)}\,\alpha\cos\Theta, \\
H_3 \;&=\; R^2 + \alpha^2\cos^2\Theta - 2\,nR, \\
\Delta \;&=\; R^2 + \alpha^2 - 2\,nR.
\end{aligned} \tag{7.28}$$

[3]This form (7.26) is obtained from (31) in [135] by changing $a \to -\alpha$. Notice that there is a typo in (36) of [135]: the minus sign of the $\mathrm{d}\phi$ term should be changed to plus. The same form of the rotating dyonic Kaluza–Klein black hole solution can be obtained from (A.1) in [138] by changing $\phi \to -\Phi$.

The (four-dimensional) physical quantities, the mass M, angular momentum J, electric charge Q, and magnetic charge P of the rotating dyonic Kaluza–Klein black hole (7.26) are respectively given by

$$M = \frac{p+q}{4}, \quad J = -\frac{\sqrt{pq}(pq+4n^2)\alpha}{4n(p+q)}, \quad Q = \sqrt{\frac{q(q^2-4n^2)}{4(p+q)}}, \quad P = \sqrt{\frac{p(p^2-4n^2)}{4(p+q)}}. \tag{7.29}$$

As a five-dimensional metric, the rod structure of the above solution (7.26) can be calculated, though in practice, it might be a tedious task. However, it is not difficult to anticipate the rod structure from the construction of the solution (7.26) in the equivalent form of [91] by using the $SL(3,R)$ generating techniques on the direct product of a flat time dimension and the Kerr solution. It was shown in [62, 92] that applying the $SL(3,R)$ generating techniques does not alter the number of the turning points, but solely alters the alternative directions of the rods. Hence we would expect that, after being brought to standard orientation, the rod structure of the solution (7.26) is similar to that of (7.6), except that the horizon rod now has two independent non-zero angular velocities. The asymptotic structure of the solution (7.26) is a twisted but finite S^1 fiber bundle over $M^{1,3}$, and the background space-time can be shown to be the self-dual Taub-NUT instanton with a flat time dimension. Hence we naturally interpret the solution (7.26), from the five-dimensional perspective, as the double-rotating Myers–Perry black hole on Taub-NUT. By taking the NUT charge to infinity, we would be able to recover the five-dimensional double-rotating Myers–Perry black hole (7.10).

“Blown up” KK black hole $\Leftrightarrow$ MP black hole

The NUT charges, which, as mentioned in section 6.3, characterize the size of the fifth dimension of black holes on Taub-NUT, are associated to the magnetic charges when these solutions are reduced to four dimensions in Kaluza–Klein theory. Taking the NUT charge to infinity in (7.26) then corresponds to taking the quantity P, and thus p to infinity. We can define $y = p\Psi$, $\tau = \sqrt{p}\,T$, and take out an overall factor p. In the limit $p \to \infty$, some factors in the metric (7.26) turn to unity, and then we obtain from (7.26) the “blown up” Kaluza–Klein black hole in the following form

$$\mathrm{d}s^2 = \frac{L_2\left(\mathrm{d}\Psi + A_2\right)^2}{L_1} - \frac{L_3\left(\mathrm{d}T + B_2\right)^2}{L_2} + L_1\left(\frac{\mathrm{d}R^2}{\Delta} + \mathrm{d}\Theta^2 + \frac{\Delta}{L_3}\sin^2\Theta\,\mathrm{d}\Phi^2\right), \tag{7.30}$$

where the one-forms A_2 and B_2 are defined as

$$\begin{aligned} A_2 &= -\frac{\sqrt{q}}{2}\left(\sqrt{(q^2-4n^2)} - \frac{q}{n}\alpha\cos\Theta\right)L_2^{-1}\mathrm{d}T \\ &\quad + \left\{\left(L_2 + \alpha^2\sin^2\Theta\right)\cos\Theta - \frac{\sqrt{q^2-4n^2}}{2n}\left(R-n+q\right)\alpha\sin^2\Theta\right\}L_2^{-1}\mathrm{d}\Phi, \\ B_2 &= -\frac{\sqrt{q}[qR - n\left(q-2n\right)]}{2nL_3}\alpha\sin^2\Theta\,\mathrm{d}\Phi, \end{aligned} \tag{7.31}$$

and the functions L_1, L_2, L_3 and Δ are given by

$$\begin{aligned} L_1 &= R + \frac{q-2n}{2} + \frac{\sqrt{q^2-4n^2}\,\alpha\cos\Theta}{2n}, \\ L_2 &= R^2 + \alpha^2\cos^2\Theta + \frac{\left(2R+q\right)\left(q-2n\right)}{2} - \frac{q\sqrt{q^2-4n^2}\,\alpha\cos\Theta}{2n}, \\ L_3 &= R^2 + \alpha^2\cos^2\Theta - 2nR, \\ \Delta &= R^2 + \alpha^2 - 2nR. \end{aligned} \tag{7.32}$$

As expected, the above blown up Kaluza–Klein black hole is equivalent to the five-dimensional Myers–Perry black hole (7.10) via the following coordinate transformations and redefinitions of parameters

$$
\begin{gathered}
R = \tfrac{1}{8}(2r^2 + a_1^2 + a_2^2 - 2m + \sqrt{2m[2m - (a_1 + a_2)^2]}), \qquad \Theta = 2\theta, \\
T = t, \qquad \Psi = \phi_1 + \phi_2, \qquad \Phi = -\phi_1 + \phi_2, \\
\alpha = \tfrac{1}{8}\,(a_1 - a_2)\sqrt{2m - (a_1 + a_2)^2}, \quad n = \tfrac{1}{8}\sqrt{2m\,[2m - (a_1 + a_2)^2]}\,. \quad (7.33)
\end{gathered}
$$

This equivalence further confirms the interpretation that the solution (7.26) is the Taub-NUT generalization of the five-dimensional Myers–Perry black hole (7.10).

Introducing extra factors to MP black hole

The above equivalence of the solutions (7.30) and (7.10) allows us to introduce some extra factors to the five-dimensional Myers–Perry black hole (7.10) to get the Myers–Perry black hole on Taub-NUT (7.26). To do this, we first write the five-dimensional Myers–Perry black hole in the following form

$$
\mathrm{d}s^2 = \frac{h_2\,(\mathrm{d}\psi + A)^2}{4h_1} - \frac{h_3\,(\mathrm{d}t + B)^2}{h_2} + h_1\left(\frac{r^2\mathrm{d}r^2}{\delta} + \frac{\mathrm{d}\theta^2}{4} + \frac{\delta}{4h_3}\sin^2\theta\,\mathrm{d}\phi^2\right), \quad (7.34)
$$

where the one-forms A and B are defined as

$$
\begin{aligned}
A &= A_t\,\mathrm{d}t + A_\phi\,\mathrm{d}\phi \\
&= -2m\,[a_1 + a_2 + (a_2 - a_1)\cos\theta]\,h_2^{-1}\mathrm{d}t + \Big\{h_2\cos\theta - \frac{1}{4}\,(a_1 - a_2)\,\big[\,(a_1 + a_2)\times \\
&\quad \big(2r^2 + {a_1}^2 + {a_2}^2 + 2m\big) + (a_1 - a_2)\,\big((a_1 + a_2)^2 - 2m\big)\cos\theta\big]\sin^2\theta\Big\}h_2^{-1}\mathrm{d}\phi\,, \\
B &= B_\phi\,\mathrm{d}\phi = -\frac{m\,(a_1 - a_2)\,(r^2 - a_1 a_2)}{2h_3}\sin^2\theta\,\mathrm{d}\phi\,, \qquad (7.35)
\end{aligned}
$$

and the functions h_1, h_2, h_3 and δ are given by

$$
h_1 = \frac{1}{2}\left(2r^2 + a_1^2 + a_2^2 + \left(a_1^2 - a_2^2\right)\cos\theta\right),
$$

$$
\begin{aligned}
h_2 &= \frac{1}{4}\Big[\left(2r^2+a_1^2+a_2^2\right)^2+2m(a_1+a_2)^2 \\
&\quad -4m\left(a_1^2-a_2^2\right)\cos\theta-(a_1-a_2)^2\left((a_1+a_2)^2-2m\right)\cos^2\theta\Big], \\
h_3 &= \frac{1}{4}\Big[\left(r^2+a_1^2+a_2^2\right)^2-2m\left(4r^2+(a_1+a_2)^2\right)-(a_1-a_2)^2\left((a_1+a_2)^2-2m\right)\cos^2\theta\Big], \\
\delta &= \left(r^2+a_1^2\right)\left(r^2+a_2^2\right)-2mr^2.
\end{aligned} \tag{7.36}
$$

This form (7.34) of the five-dimensional Myers–Perry black hole is equivalent to (7.10) by doing the substitution ($\psi \to \phi_1+\phi_2, \phi \to -\phi_1+\phi_2, \theta \to \frac{\theta}{2}$).

We can see that the five-dimensional Myers–Perry black hole (7.34) is equivalent to the blown up Kaluza–Klein black hole (7.30) by the following coordinate transformations

$$
\begin{gathered}
R=\tfrac{1}{8}\left(2r^2+a_1^2+a_2^2-2m+\sqrt{2m[2m-(a_1+a_2)^2]}\right), \qquad \Theta=\theta, \\
T=t, \qquad \Psi=\psi, \qquad \Phi=\phi.
\end{gathered} \tag{7.37}
$$

Recall that (7.30) is obtained from (7.26) by taking the limit $p\to\infty$. If we recover those factors of (7.26) that turn to unity in the limit $p\to\infty$, add them back to (7.30), and perform the transformation (7.37), we can introduce several extra factors to the five-dimensional Myers–Perry black hole (7.34) to get the Myers–Perry black hole on Taub-NUT (7.26). The form of solution that we obtain is

$$
\begin{aligned}
\mathrm{d}s^2 &= \frac{h_2k_1\left(\mathrm{d}\psi+k_2A_t\mathrm{d}t+k_3A_\phi\mathrm{d}\phi\right)^2}{4h_1}-\frac{h_3k_4\left(\mathrm{d}t+k_5B\right)^2}{h_2} \\
&\quad +h_1k_6\left(\frac{r^2\mathrm{d}r^2}{\delta}+\frac{\mathrm{d}\theta^2}{4}+\frac{\delta}{4h_3}\sin^2\theta\,\mathrm{d}\phi^2\right).
\end{aligned} \tag{7.38}
$$

The extra factors $k_{1\ldots6}$ just introduced contain a new parameter, say p. When $p\to\infty$, all these extra factors turn to unity, and then we recover the five-dimensional Myers–Perry black hole. The above form is equivalent to the form (7.26), which makes the interpretation of the latter as the Myers–Perry black hole on Taub-NUT manifest. Since these extra factors are quite complicated, we do not present them explicitly. In the special case when $a_2 = -a_1$, the extra factors in (7.38) are very simple. Only two of them are different from unity, and in this case, we simply recover the form (7.21).

7.6 Discussion

In this chapter, we have studied the intimate connections between black holes on Taub-NUT and Kaluza–Klein black holes. It was shown that, they can be actually obtained from the same solution, but viewed in different perspectives (of course, different identifications on coordinates may be made in different perspectives). The solution (7.26), if viewed from a five-dimensional perspective, describes the double-rotating black hole on Taub-NUT; if viewed from a four-dimensional perspective, describes the general rotating dyonic Kaluza–Klein black hole. The deep connections between these black holes suggest the possibility that five-dimensional black holes might appear as realistic objects [142]. If we are not close enough to the black hole, we cannot tell whether it is four or five-dimensional!

Black rings on Taub-NUT in vacuum gravity, Einstein–Maxwell theory, and minimal supergravity have also been constructed [62, 63, 93, 94, 98, 99]. They have very interesting interpretations when dimensionally reduced to four dimensions in Kaluza–Klein theory, though in general, singular objects might appear. Multi-black holes/black rings on Taub-NUT may also be constructed, and interpreted in Kaluza–Klein theory.

Chapter 8

Conclusion

In this thesis, we have studied stationary vacuum black hole solutions in five dimensions with $\mathbb{R} \times U(1)^2$ isometry, with a focus on the problem of classification and construction of these solutions. More specifically:

1. We developed a stronger version of the rod structure formalism (chapter 3), which allows us to analyze and classify black holes in five dimensions with $\mathbb{R} \times U(1)^2$ isometry. The rod structure formalism encodes much information of a space-time, and helps one to gain deep insights into the geometrical and topological properties of the space-time considered.

2. We constructed the black lens solutions with horizon topology $L(n, 1)$ in five-dimensional asymptotically flat space-times (chapter 4). Unfortunately, it was found that there are either conical or naked singularities present in such space-times even when we turn on all the rotations.

3. We classified, in terms of the rod structure, gravitational instantons with $U(1) \times U(1)$ isometry (chapter 5), which can serve as spatial backgrounds of five-dimensional black holes. It was found that the rod structure alone cannot uniquely determine a gravitational instanton. We also listed down some rod structures that would correspond to new gravitational instantons.

4. We classified and constructed black holes in five dimensions on gravitational instantons with $U(1) \times U(1)$ isometry and up to two turning points (chapter 6). These black holes have various asymptotic geometries other than $M^{1,4}$ or $M^{1,3} \times S^1$. In particular, black holes on Euclidean Kerr, Eguchi–Hanson and Taub-bolt instantons were constructed and their properties were studied.

5. We interpreted the rotating dyonic Kaluza–Klein black holes, when appropriately lifted to five dimensions, as Myers–Perry black holes on Taub-NUT (chapter 7). In particular, the Ishihara–Matsuno black hole and Wang's rotating black hole were shown to be equivalent to the static magnetic and static dyonic Kaluza–Klein black holes, if lifted to five dimensions, respectively.

To a certain extent, the classification of vacuum black holes in five dimensions with $\mathbb{R} \times U(1)^2$ isometry in terms of the rod structure formalism discussed in this thesis is close to being a complete one. In particular, the rod structure of a solution with $\mathbb{R} \times U(1)^2$ isometry determines the global topology of the space-time. In the asymptotically $M^{1,4}$ or $M^{1,3} \times S^1$ space-times, together with the angular momenta, the rod structure can uniquely determine a solution [38, 39]; and we suspect this is true in all the space-times with asymptotic geometries studied in this thesis, if these asymptotic geometries are specified in some appropriate way. Of course, a rigorous proof of this conjecture needs to be worked out.

The problem of the construction of all the vacuum black holes in five dimensions with $\mathbb{R} \times U(1)^2$ isometry is still open. Firstly, it is still not known whether all the rod structures that are allowed in our analysis can be realized in explicit solutions. Even in the asymptotically flat case, the completely regular black lenses have not been found, neither has their existence been disproved. Secondly, although a proof might be possible in the future such that all black holes in five dimensions with $\mathbb{R} \times U(1)^2$ isometry can be in principle generated by applying the inverse scattering method, the labor needed to carry out the explicit constructions may grow so rapidly, as the number of solitons involved grows, that these constructions become practically impossible.

We end this thesis by giving some possible extensions of the present work as well as some open problems for the future:

1. To complete the project of this thesis, i.e., to construct all black holes in five dimensions with $\mathbb{R} \times U(1)^2$, and gravitational instantons with $U(1) \times U(1)$ isometry. We are very aware that there are actually more unknown solutions within our classification schemes than the known ones. The particularly interesting ones include the double-rotating black rings on Taub-NUT, and double-rotating black holes on Euclidean Kerr and Taub-bolt instantons.

2. To prove the (non-)uniqueness properties of the solutions studied in this thesis. One might expect that uniqueness theorems can be proved along the lines of Hollands and Yazadjiev [38, 39].

3. To generalize the stronger version of rod structure formalism to black holes in D dimensions with $\mathbb{R} \times U(1)^{D-3}$ isometry and their background spaces for $D > 5$ [35, 36, 39, 40]. The regular background spaces will be $(D-1)$-dimensional generalizations of gravitational instantons with $U(1)^{D-3}$ isometry. In fact, a number of five-dimensional gravitational instantons within such a class have been found, to mention a few [80, 81, 143], but yet to be analyzed in terms of the rod structure. As already mentioned, necessary regularity conditions at the turning points of these space(-times) must be carefully treated [39].

4. To classify and construct black holes and gravitational instantons in higher dimensions with fewer isometries than mentioned above. These solutions might be treated by generalizing the rod-structure formalism to the domain structure [144]. For the case of five-dimensional stationary black holes with $\mathbb{R} \times U(1)$ isometry, the possible horizon topologies have recently been classified in [145]. We note that the inverse scattering method no longer applies in the construction of these solutions, although the $SL(3, R)$ solution-generating technique might still be applicable.

5. To classify and construct black holes in five dimensions with $\mathbb{R} \times U(1)^2$ isometry in more complicated contexts. In fact, most of the currently known exact black hole solutions in five dimensions in the literature possess the isometry group $\mathbb{R} \times U(1)^2$, including, in particular, those in Maxwell–Einstein theory and in minimal supergravity. Some recent progresses have already been made in this direction. The generalized Weyl formalism has been generalized to Einstein–Gauss–Bonnet theory [146], and the inverse scattering method has

been generalized to construct solutions in five-dimensional minimal supergravity [147].

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Appendix A

ISM construction of black lenses

In this appendix, we sketch how to construct the double-rotating black lens solution using the inverse scattering method. We present here the seed solution and some necessary BZ operations. It is not difficult for the reader to follow the general procedure described in section 3.2.1 to carry out the explicit constructions. The single-rotating black lens (4.21) and the static black lens (4.1) can be constructed as special cases that will be described.

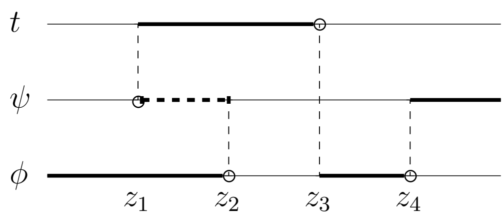

Figure A.1: The rod structure of the seed for the double-rotating black lens.

We can use the seed that was used to generate the Emparan–Reall black ring, whose rod structure is shown in Fig. A.1. We perform the following BZ operations:

1. remove a soliton at each of z_1, z_2, z_3 and z_4, with BZ vectors $(0, 1, 0)$, $(0, 0, 1)$, $(1, 0, 0)$ and $(0, 0, 1)$ respectively,
2. add the same soliton back at each of z_1, z_2, z_3 and z_4, with BZ vectors $(C_1, 1, 0)$, $(0, C_2, 1)$, $(1, 0, C_3)$ and $(0, C_4, 1)$ respectively.

The seed and modified seed solutions are:

$$
\begin{aligned}
g_0 &= \begin{bmatrix} -\frac{\mu_1}{\mu_3} & 0 & 0 \\ 0 & \frac{\mu_2}{\mu_1}\mu_4 & 0 \\ 0 & 0 & \frac{\rho^2}{\mu_2}\frac{\mu_3}{\mu_4} \end{bmatrix}, \\
\tilde{g}_0 &= g_0 \begin{bmatrix} -\frac{\mu_3^2}{\rho^2} & 0 & 0 \\ 0 & -\frac{\mu_1^2}{\rho^2} & 0 \\ 0 & 0 & (-\frac{\mu_2^2}{\rho^2}) \times (-\frac{\mu_4^2}{\rho^2}) \end{bmatrix} = \begin{bmatrix} \frac{\mu_1\mu_3}{\rho^2} & 0 & 0 \\ 0 & -\frac{\mu_1\mu_2\mu_4}{\rho^2} & 0 \\ 0 & 0 & \frac{\mu_2\mu_3\mu_4}{\rho^2} \end{bmatrix}. \\
e^{2\nu_0} &= \frac{\mu_2\mu_4 R_{13}R_{12}R_{14}R_{23}R_{34}}{\mu_1 R_{24}^2 R_{11}R_{22}R_{33}R_{44}}.
\end{aligned}
\tag{A.1}
$$

In the BZ operations we describe above, we remove and add back the four solitons at one time. This is, however, not necessary. We can instead do the following BZ operations (see Fig. A.2):

1. add a soliton at z_1 with BZ vector $(1, 0, 0)$;
2. add an anti-soliton at z_1 with BZ vector $(1, C1, 0)$;

3. remove a soliton at each of z_2 and z_4, with BZ vectors $(0, 0, 1)$, and add a soliton at z_3 with BZ vector $(0, 0, 1)$;

4. add back a soliton at each of z_2 and z_4, with BZ vectors $(0, C_2, 1)$ and $(0, C_4, 1)$ respectively; and add an anti-soliton at z_3 with BZ vector $(C_3, 0, 1)$.

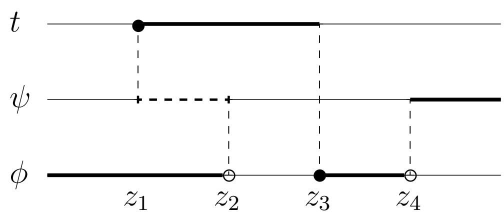

Figure A.2: The rod structure of an alternative seed for the double-rotating black lens.

These BZ operations have the advantage that it allows us to generate the Emparan–Reall black ring after the first two steps. So we effectively eliminate the first turning point in the seed, and can cast the solution to C-metric coordinates. This leads to significant simplifications, since all subsequent BZ operations can be done in C-metric coordinates. At this point, ϕ is still not mixed with t and ψ. Step 3 can then be easily carried out by multiplying $g_{\phi\phi}$ by some simple factors. After eliminating the Dirac–Misner singularity, the final solution is then the double-rotating black lens with two independent angular momenta.

The Pomeransky–Sen'kov black ring [49] can be obtained by setting $C_4 = 0$; while the single-rotating black lens (4.21) can be obtained by setting $C_3 = 0$, and if we further set $z_1 = z_2$, the static black lens (4.1) is obtained.

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